

RIGOROUS RESULTS IN EXISTENCE AND SELECTION OF SAFFMAN-TAYLOR FINGERS BY KINETIC UNDERCOOLING

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ABSTRACT. The selection of Saffman-Taylor fingers by surface tension has been extensively investigated. Here in this paper we are concerned about existence and selection of steadily translating symmetric finger solutions in a Hele-Shaw cell by small but non-zero kinetic undercooling (ϵ^2). We rigorously conclude that for relative finger width λ in the range $[\frac{1}{2}, \lambda_m]$, with $\lambda_m - \frac{1}{2}$ small, symmetric finger solutions exist in the asymptotic limit of undercooling $\epsilon^2 \rightarrow 0$ if the Stokes constant for a relatively simple nonlinear differential equation is zero. This Stokes constant S depends on the parameter $a \equiv \frac{2\lambda-1}{(1-\lambda)}\epsilon^{-\frac{4}{3}}$ and earlier calculations have shown this to be zero for a discrete set of values of a . While this result is similar to that obtained previously for Saffman-Taylor fingers by surface tension, the analysis for the problem with kinetic undercooling exhibits a number of subtleties as pointed out by Chapman and King [3]. The main subtlety is the behavior of Stokes lines at the finger tip, which leads to existence of possible non-analytic fingers by kinetic undercooling, while previous results show Saffman-Taylor fingers by surface tension must be analytic.

1. INTRODUCTION

1.1. Mathematical Formulation. We consider the problem of fingers in a Hele-Shaw cell for small kinetic undercooling. The gap-averaged velocity (u, v) in the (x, y) plane in the region outside the finger satisfies

$$(1.1) \quad (u, v) = -\frac{b^2}{12\mu} \nabla p \equiv \nabla \phi,$$

where b is the gap width, μ the viscosity of the more viscous fluid and p denotes the pressure. Incompressibility of fluid flow implies zero divergence of fluid velocity,

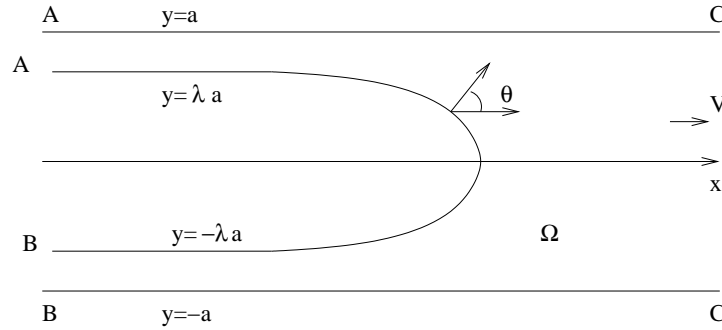


FIGURE 1. A translating finger in the Hele-Shaw cell

implying that the velocity potential ϕ satisfies

$$(1.2) \quad \Delta\phi = 0 \text{ for } (x, y) \in \Omega$$

The boundary conditions on the side walls are

$$(1.3) \quad v = 0 \text{ when } y = \pm a$$

Far-ahead of the finger, in the laboratory frame, we have

$$(1.4) \quad (u, v) = V\hat{x} + O(1)$$

where \hat{x} is a unit vector in the x -direction (along the Hele-Shaw channel) and V is the constant displacement rate of the fluid far away.

We set the channel half-width $a = 1$, displacement rate $V + 1$, which corresponds to non-dimensionalizing all lengths by a and velocities by V (consequently time is measured in units of a/V).

The kinematic condition for a steady finger is:

$$(1.5) \quad \frac{\partial\phi}{\partial n} = U \cos\theta;$$

where θ is the angle between the interface normal and the positive x -axis (see Fig. 1) and U is the speed of the finger.

The kinetic undercooling condition

$$(1.6) \quad \phi = cv_n = cU \cos\theta;$$

where c is the kinetic undercooling parameter.

We can set $a = 1$ and $V = 1$ as this corresponds to nondimensionalizing all lengths by a and velocities by V .

By integrating (1.2) in the domain Ω , the finger width λ is related to U as follows:

$$(1.7) \quad \lambda = \frac{1}{U};$$

In a frame moving with the steady symmetric finger, the condition (1.5) transforms, without loss of generality, into

$$(1.8) \quad \psi = 0;$$

where ψ is the stream function (harmonic conjugate of ϕ) so that $W = \phi + i\psi$ is an analytic function of $z = x + iy$.

The nondimensional kinetic undercooling condition (1.6) in the moving frame becomes

$$(1.9) \quad \phi + \frac{1}{\lambda}x = \frac{c}{\lambda} \cos\theta.$$

On the side walls, (1.3) implies that

$$(1.10) \quad \psi = \pm \left[\frac{1}{\lambda} - 1 \right] \text{ on } y = -1 \text{ and } y = 1 \text{ respectively}$$

while the far field condition as $z \rightarrow +\infty$ in the finger frame is

$$(1.11) \quad W \sim - \left[\frac{1}{\lambda} - 1 \right] z + O(1);$$

We consider the conformal map $z(\xi)$ of the cut upper-half- ξ -plane, as shown in Figure 2, to the flow domain Ω in Figure 1. The real ξ axis corresponds to the finger boundary, with $\xi = -\infty, +\infty$ corresponding to the finger tails at $z = -\infty + i\lambda, z =$

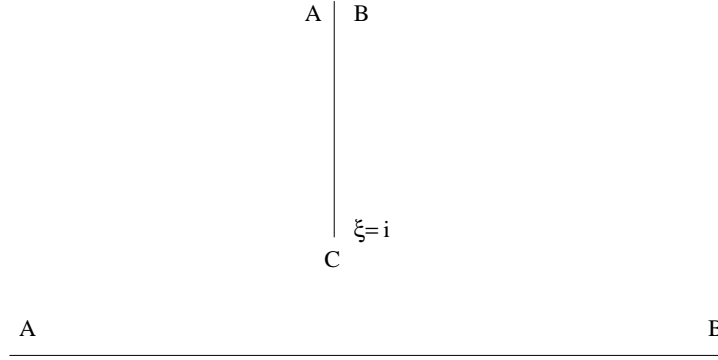


FIGURE 2. Upper half ξ -plane with a cut on imaginary axis from $\xi = i$ to $i\infty$

$-\infty - \lambda i$ respectively. The two sides of the cut correspond to the two side walls respectively. It is easily seen that the complex potential $W(\xi)$ is given by:

$$(1.12) \quad W(\xi) = \frac{1-\lambda}{\pi\lambda} \ln(\xi^2 + 1) + \text{constant}$$

We define $F(\xi)$ so that

$$(1.13) \quad z(\xi) = -\frac{1}{\pi} \ln(\xi - i) - \frac{1-2\lambda}{\pi} \ln(\xi + i) - i\lambda - \frac{2-2\lambda}{\pi} F(\xi);$$

It follows that F , as defined above, is analytic in the entire upper half ξ -plane (Tanveer 2000). The kinetic undercooling condition (1.9) translates into requiring that on the real ξ axis:

$$(1.14) \quad \text{Re } F = -\epsilon^2 \frac{\text{Im}(F' + H)}{|F' + H|};$$

where

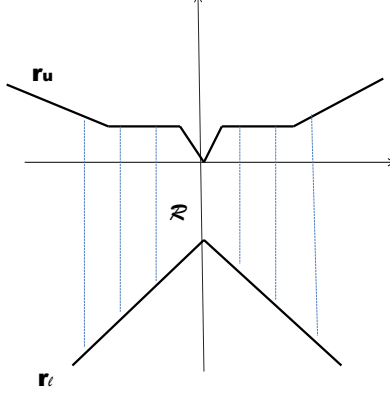
$$(1.15) \quad H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1}, \text{ with } \gamma = \frac{\lambda}{1-\lambda}, \epsilon^2 = \frac{c\pi}{2\lambda(1-\lambda)};$$

For zero kinetic undercooling $\epsilon = 0$, it follows that $F = 0$; this corresponds to what is usually referred to in the literature as Saffman and Taylor solutions (Zhuravlev (1956), Saffman & Taylor (1958)). This form a family of exact solutions for symmetric fingers with arbitrary width of finger $\lambda \in (0, 1)$. The Saffman-Taylor solutions, in our formulation, correspond to the conformal map

$$(1.16) \quad z_0(\xi) = -\frac{1}{\pi} \ln(\xi - i) - \frac{1-2\lambda}{\pi} \ln(\xi + i) - i\lambda;$$

This is easily seen to be univalent since the boundary correspondence is one to one. In particular, the finger shape can be explicitly described by $\text{Re } z$ as a function of $\text{Im } z$.

For non-zero kinetic undercooling, there has been some research on the problem [?, 9, 12, 13, 3, 23]. Chapman and King's (2003) numerical and asymptotic work suggested that solutions exist only for infinitely many isolated values of finger width which are bigger than $\frac{1}{2}$. The aim of this paper is to rigorously confirm these results. When the relative finger width λ is smaller than $\frac{1}{2}$, no steadily translating

FIGURE 3. Region \mathcal{R} in complex ξ plane

symmetric finger solution exists [23].

It is to be noted that there is a huge amount of research on the famous finger problem with surface tension [5, 15, 18, 19, 2, 22, 20].

1.2. Notations.

Definition 1.1. Let \mathcal{R} be an open connected (see Figure 3) region on complex ξ plane bounded by lines r_u and r_l defined as follows:

$$r_u = r_{u,1} \cup r_{u,2} \cup r_{u,3} \cup r_{u,4} \cup r_{u,5} \cup r_{u,6}$$

$$r_l = \{\xi : \xi = -bi + re^{-i(\varphi_0 + \mu)}\} \cup \{\xi : \xi = -bi + re^{i(\pi + \varphi_0 + \mu)}\}$$

where $0 < b < \text{Min}(1, \gamma)$, $0 < \varphi_0$, $\mu < \frac{\pi}{2}$ with $\varphi_0 + \mu < \frac{\pi}{2}$ and

$$r_{u,1} = \{\xi : \xi = \nu_1 i - R + re^{i(\pi - \varphi_0)}, 0 \leq r < \infty\},$$

$$r_{u,2} = \{\xi : \text{Im } \xi = \nu_1, -R \leq \text{Re } \xi \leq -\nu_1\},$$

$$r_{u,3} = \{\xi : \xi = re^{3\pi i/4}, 0 \leq r \leq \sqrt{2}\nu_1\},$$

$$r_{u,4} = \{\xi : \xi = re^{\pi i/4}, 0 \leq r \leq \sqrt{2}\nu_1\}$$

$$r_{u,5} = \{\xi : \text{Im } \xi = \nu_1, \nu_1 \leq \text{Re } \xi \leq R\},$$

$$r_{u,6} = \{\xi : \xi = \nu_1 i + R + re^{i\varphi_0}, 0 \leq r < \infty\}$$

where $R > 0$ is large enough and $0 < \nu_1$ is small enough so that Lemmas in the appendix hold.

Remark 1.2. $1 - b$, and φ_0 are chosen independent of ϵ . There are some restrictions imposed on their values in order that certain Lemmas in the appendix are ensured.

Definition 1.3. $\mathcal{R}^- = \mathcal{R} \cap \{\text{Re } \xi < 0\}$; $\mathcal{R}^+ = \mathcal{R} \cap \{\text{Re } \xi > 0\}$

Let $0 < \tau < 1$ be independent of ϵ , $\nu > 0$ be a small number, and $B_\nu = \{\xi : |\xi| < \nu\}$. We introduce spaces of functions:

Definition 1.4. Define

$$\begin{aligned} \mathbf{A}_0^- &= \{F : F(\xi) \text{ analytic in } \mathcal{R}^- \text{ and continuous in } \overline{\mathcal{R}^-}, \\ &\quad \text{with } \sup_{\xi \in \overline{\mathcal{R}^-}} |(\xi - 2i)^\tau F(\xi)| < \infty \\ &\quad \text{and } \sup_{\xi \in \overline{\mathcal{R}^-}} \left| \frac{F'(\xi) - F'(0)}{\sqrt{\xi}} \right| < \infty\} \\ \|F\|_0^- &:= \sup_{\xi \in \overline{\mathcal{R}^-}} |(\xi - 2i)^\tau F(\xi)| + \epsilon^2 \sup_{\xi \in \overline{\mathcal{R}^-} \cap B_\nu} \left| \frac{F'(\xi) - F'(0)}{\sqrt{\xi}} \right| + |F'(0)| \end{aligned}$$

Remark 1.5. \mathbf{A}_0^- is a Banach space. If $F \in \mathbf{A}_0^-$, then F satisfies property:

$$(1.17) \quad F(\xi) \sim O(\xi^{-\tau}), \text{ as } |\xi| \rightarrow \infty, \xi \in \mathcal{R}^-.$$

In Definition 1.4, replacing \mathcal{R}^- with \mathcal{R}^+ we can define \mathbf{A}_0^+ and $\mathbf{A} \equiv \mathbf{A}_0 \equiv \mathbf{A}_0^- \cap \mathbf{A}_0^+$ with norm $\|F\|_0 = \|F\|_0^- + \|F\|_0^+$.

Remark 1.6. Locally in a neighborhood of $\xi = 0$, \mathbf{A}_0 contains functions which are analytic at $\xi = 0$, it also contains functions $F(\xi)$ which are NOT analytic at $\xi = 0$ such as $F(\xi) = C\xi^{k+3/2}$, $k = 0, 1, 2, \dots$.

Definition 1.7. Let \mathcal{D} be any connected set in the complex ξ -plane; we introduce norms: $\|F\|_{k,\mathcal{D}} := \sup_{\xi \in \mathcal{D}} |(\xi - 2i)^{k+\tau} F(\xi)|$, $k = 0, 1, 2$.

Definition 1.8. Let $\delta > 0$ be a constant, define space:

$$\mathbf{A}_{0,\delta}^- = \{f : f \in \mathbf{A}^-, \|f\|_0^- \leq \delta\}.$$

$\mathbf{A}_{0,\delta}^-$ and $\mathbf{A}_{0,\delta}$ can be defined similarly.

Remark 1.9. For λ in a compact subset of $(0, 1)$, for sufficiently small enough δ , if $F \in \mathbf{A}_{0,\delta}$, then the mapping $z(\xi)$ given by (1.13) is univalent. See Theorem 1.5 in [22].

The problem of determining a smooth steady symmetric finger is equivalent to finding function analytic F in the upper-half ξ plane, which is \mathbf{C}^1 in its closure, *i.e.* continuous derivatives on $\text{Im } \xi = 0$ and is required to satisfy the following conditions:

Condition (i): $F(\xi)$ satisfies (1.14) on the real ξ axis.

Condition (ii):

$$(1.18) \quad F(\xi) \rightarrow 0 \text{ as } \xi \rightarrow \pm\infty;$$

Condition (iii)(symmetry condition):

$$(1.19) \quad \text{Re } F(-\xi) = \text{Re } F(\xi), \text{ Im } F(-\xi) = -\text{Im } F(\xi) \text{ for } \xi \text{ real};$$

Remark 1.10. If $F \in \mathbf{A}_0$, and satisfies symmetry condition (iii) on the real axis, then it follows from Cauchy integral formula on the imaginary ξ axis segment $-b < \text{Im } \xi < 0$, starting at $\xi = 0$ that $\text{Im } F(\xi) = 0$. From Schwartz reflection principle, $F(\xi) = [F(-\xi^*)]^*$, so $\|F\|_0^- = \|F\|_0^+$. Conversely, if $F \in \mathbf{A}_0^-$ and satisfies $\text{Im } F(\xi) = 0$ for the imaginary ξ axis segment $-b < \text{Im } \xi < 0$, then $F(\xi) = [F(-\xi^*)]^*$ extends F to the right half of \mathcal{R} and $\|F\|_0^- = \|F\|_0^+$ and (1.19) is then automatically satisfied.

Finger problem: *The problem tackled in this paper will be to find function F analytic in $\{Im \xi > 0\} \cup \mathcal{R}$ so that $F \in \mathbf{A}_{0,\delta}$ for some τ fixed in $(0, 1)$, δ small, so that conditions (i) and (iii) on the real axis are satisfied.*

1.3. Main Results. Similar to the finger problem with surface tension (Combescot et al 1986, 1988, Tanveer 1987, [22]), the formal strategy of calculation of finger width involves analytic continuation of equation in an inner neighborhood of turning points in the complex plane and ignoring integral contribution and other terms that are formally small. The problem of determining a smooth steady finger is reduced to determining eigenvalues a so that $G(y)$ is a solution to

$$(1.20) \quad \frac{dG}{dy} + \frac{1}{yG^2} = -y - \frac{a}{2^{1/3}y}$$

satisfying the condition

$$(1.21) \quad yG(y) \rightarrow 1 \text{ as } y \rightarrow \infty \text{ for } \arg y \in [0, \pi/4);$$

and in addition satisfying:

$$(1.22) \quad Im G = 0, \text{ for large enough } y \text{ on the positive real axis.}$$

The finger width λ is related to a through

$$(1.23) \quad a = \frac{2\lambda - 1}{1 - \lambda} \epsilon^{-4/3};$$

We introduce additional change in variables:

$$(1.24) \quad \eta = \frac{2}{3}y^3, \quad \psi(\eta) = 1 - yG(y);$$

then (1.20) becomes:

$$(1.25) \quad \frac{d\psi}{d\eta} + \psi = -\frac{1}{3\eta} - \frac{1}{3\eta}\psi + \frac{a}{6^{2/3}\eta^{2/3}} + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n$$

and matching conditions as $|\eta| \rightarrow \infty$:

$$(1.26) \quad \psi(\eta, a) \sim 0, 0 \leq \arg \eta < \frac{3\pi}{4};$$

One of the theorems proved in this paper is

Theorem 1.11. (1) *There exists sufficiently large enough $\rho_0 > 0$ such that (1.25) with condition (1.26) has unique analytic solution $\psi_0(\eta, a)$ in the region $|\eta| > \rho_0, \arg \eta \in [0, \frac{3\pi}{4}]$.*

(2) *Further on the real η axis as $\eta \rightarrow \infty$*

$$(1.27) \quad Im \psi_0(\eta, a) \sim \tilde{S}(a)e^{-\eta};$$

(3) *Further $Im \psi_0(\eta, a) = 0$ for real η and $\eta > \rho_0$ iff $\tilde{S}(a) = 0$.*

we will not compute the Stokes constants $\tilde{S}(a)$, Chapman and King (2003) numerical computation indicates that there exist a discrete set $\{a_n\}$ so that $\tilde{S}(a_n) = 0, \tilde{S}'(a_n) \neq 0$. It is to be noted that the theory of exponential asymptotics for general nonlinear ordinary differential equations (Costin, 1998) makes it possible to rigorously confirm these calculations to within small error bounds, though this analysis is yet to be carried out for this problem.

The primary result of this paper for the finger problem is the following:

Theorem 1.12. *In a range $\frac{1}{2} \leq \lambda \leq \lambda_m$, δ and $\lambda_m - \frac{1}{2}$ small (but independent of ϵ) so that (2.24) holds, the following statement hold for all sufficiently small ϵ : For each $\beta_0 \in \{a_n\}$ for which the Stokes constant $\tilde{S}(\beta_0) = 0$ in Theorem 1.11, if $\tilde{S}'(\beta_0) \neq 0$, there exists an analytic function $\beta(\epsilon^{2/3})$ with $\lim_{\epsilon \rightarrow 0} \beta(\epsilon^{2/3}) = \beta_0$ so that if*

$$(1.28) \quad \frac{2\lambda - 1}{1 - \lambda} = \epsilon^{4/3} \beta(\epsilon^{2/3}),$$

then there exists a solution of the Finger problem $F \in \mathbf{A}_{0,\delta}$. Hence for small ϵ ,

$$(1.29) \quad \frac{2\lambda - 1}{1 - \lambda} = \epsilon^{4/3} \left(\beta_0 + \beta_1 \epsilon^{2/3} + \beta_2 \epsilon^{4/3} + \dots \right);$$

The proof of this theorem is not given until the end of §4, after many preliminary results. Our solution strategy consists of two steps:

(a) Relaxing the symmetry condition $F(\xi) = [F(-\xi^*)]^*$ on the imaginary axis interval $-b < \text{Im } \xi < 0$, (*i.e.* $\text{Im } F = 0$ is relaxed on that $\text{Im } \xi$ -axis segment), we prove the existence of solutions to an appropriate problem in the half strip \mathcal{R}^- for any λ belonging to a compact subset of $(0, 1)$, for all sufficiently small ϵ . There is no restriction on λ otherwise.

(b) The symmetry condition is then invoked to determine restriction on λ that will guarantee existence of solution to the Finger problem. In this part, we restrict our analysis to $\lambda \in [\frac{1}{2}, \lambda_m]$.

In Section 2, we prove equivalence of the finger problem to a set of two problems (*Problem 1 and Problem 2*) in a complex strip domain. Problem 1 is to solve an integro-differential equation for F in a Banach space \mathbf{A}_0 . By deforming the contour of integration for the integral term in Problem 1, we obtain Problem 2. By relaxing symmetry condition on an $\text{Im } \xi$ axis segment, we derive a Half Problem in the left half strip \mathcal{R}^- .

In Section 3, by constructiong a normal sequence, the existence of solutions to the Half problem is proved for λ in a compact subset of $(0, 1)$ for all sufficiently small ϵ . In Section 4, we carry out step (b) in our solution strategy. By introducing suitably scaled dependent and independent variables in a neighborhood of a turning point, we formulate the inner problem. For the leading order inner-equation, the form of exponentially small terms are obtained and Theorem 1.11 proved. For the full problem, using implicit function theorem, it is argued that for small ϵ , there exists a discrete set of analytic functions $\beta(\epsilon^{2/3})$ so that if $a = \beta(\epsilon^{2/3})$, then $\text{Im } F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$. This implies that symmetry condition (1.19) is satisfied; hence Theorem 1.12 follows.

While the main result and strategy of proof are similar to that in [22] for Saffman-Taylor fingers by surface tension, the analysis for the problem with kinetic undercooling exhibits a number of subtleties as pointed out by Chapman and King [3]. The main subtletiy is the behaviour of Stokes lines at the finger tip, which leads to existence of possible non-analytic fingers by kinetic undercooling (see section 3), while previous results [20] show Saffman-Taylor fingers by surface tension must be analytic.

2. FORMULATION OF EQUIVALENT PROBLEMS

In this particular section, we are going to formulate Problem 1 and Problem 2 which will be proved to be equivalent to Finger Problem defined in Section 1. We

then formulate a half problem in terms of an integro-differential equation in \mathcal{R}^- , but relax the symmetry condition $\text{Im } F = 0$ on the imaginary ξ -axis segment $(-ib, 0)$, which follows from (1.19) (see Remark 1.10). Problem 2, unlike Problem 1, involves nonlocal quantities (I_2 for instance) with integration paths outside domain \mathcal{R} . Hence, when symmetry is relaxed for F in the half problem, singularities at the origin from nonlocal contributions can be estimated conveniently in terms of F from the domain \mathcal{R}^- . In this section, as well as the next, we will restrict λ to a compact subset of $(0, 1)$ so that all the constants appearing in all estimates are independent of λ .

2.1. Formulation of Problem 1.

Definition 2.1. Define

$$(2.1) \quad \bar{H}(\xi) = [H(\xi^*)]^* = \frac{\xi - i\gamma}{\xi^2 + 1};$$

$$(2.2) \quad \bar{F}(\xi) = [F(\xi^*)]^*;$$

Remark 2.2. If F is analytic in domain \mathcal{D} containing real axis with property (1.17), then \bar{F} is analytic in \mathcal{D}^* and $\bar{F}(\xi) = F^*(\xi)$ for ξ real and $\bar{F}(\xi) \sim O(\xi^{-\tau})$, as $|\xi| \rightarrow \infty, \xi \in \mathcal{D}^*$; $\bar{F}'(\xi) \sim O(\xi^{-1-\tau})$, $\bar{F}''(\xi) \sim O(\xi^{-2-\tau})$ as $|\xi| \rightarrow \infty, \xi \in \mathcal{D}^*$, where \mathcal{D}^* is any angular subset of \mathcal{D} and superscript $*$ denotes conjugate domain obtained by reflecting about the real axis.

Definition 2.3. Let \mathcal{D} be a connected set; for any two functions f, g with derivative existing in \mathcal{D} and small enough $\|f'\|_{1,\mathcal{D}}$ and $\|g'\|_{1,\mathcal{D}}$ so that $f' + H \neq 0, g' + \bar{H} \neq 0$ in \mathcal{D} , we define operator G so that

$$(2.3) \quad G(f, g)[t] := \frac{1}{(f'(t) + H(t))^{1/2}(g'(t) + \bar{H}(t))^{1/2}} \times [(f'(t) + H(t)) - (g'(t) + \bar{H}(t))]$$

Remark 2.4. If $F \in \mathbf{A}$ and $F' + H \neq 0$ in \mathcal{R} , then $G(F, \bar{F})(t)$ is analytic in $\mathcal{R} \cap \mathcal{R}^*$, since in that case $\bar{F}' + \bar{H} \neq 0$ in \mathcal{R}^* .

Lemma 2.5. Let \mathcal{D} and f, g be as in definition 2.3. If each of $\text{dist}(\mathcal{D}, -i)$, $\text{dist}(\mathcal{D}, i)$, $\text{dist}(\mathcal{D}, -i\gamma)$, and $\text{dist}(\mathcal{D}, i\gamma)$ are greater than 0 and independent of ϵ , as $\epsilon \rightarrow 0$, then we have for small enough $\|f'\|_{1,\mathcal{D}}, \|g'\|_{1,\mathcal{D}}$,

$$(2.4) \quad \|G(f, g)\|_{0,\mathcal{D}} \leq \frac{C(1 + \|g'\|_{1,\mathcal{D}} + \|f'\|_{1,\mathcal{D}})}{(K_1 - \|f'\|_{1,\mathcal{D}})(K_1 - \|g'\|_{1,\mathcal{D}})}$$

where

$$(2.5) \quad \begin{aligned} 0 < H_m &\leq \inf_{\mathcal{D}} \{|(t - 2i)H(t)|, |(t - 2i)\bar{H}(t)|\}; \\ K &\geq \sup_{\mathcal{D}} |t - 2i|^{-\tau} > 0; \\ K_1 &= \frac{H_m}{K} > 0; \end{aligned}$$

Constants C , K and K_1 are independent of ϵ and λ , since λ is in a compact subset of $(0, 1)$.

Proof. Without ambiguity, the norms $\|\cdot\|$ denoted in this proof refer to $\|\cdot\|_{\mathcal{D}}$, where sup is over the domain \mathcal{D} .

Using (1.15), (2.1):

$$\begin{aligned} \sup\{|t - 2i|^2|\bar{H}(t) - H(t)|\} &\leq C \\ \sup|(t - 2i)H| &\leq C, \sup|(t - 2i)\bar{H}| \leq C, \end{aligned}$$

where C is made independent of ϵ and λ for λ in a fixed compact subset of $(0, 1)$. We also have

$$\begin{aligned} |(f'(t) + H(t))| &\geq [H_m|t - 2i|^{-1} - \|f'\|_1|t - 2i|^{-1-\tau}] \\ &\geq (H_m - K\|f'\|_1)|t - 2i|^{-1} \\ &\geq C(K_1 - \|f'\|_1)|t - 2i|^{-1}; \\ (2.6) \quad |(g'(t) + \bar{H}(t))| &\geq [H_m|t - 2i|^{-1} - \|g'\|_1|t - 2i|^{-1-\tau}] \\ &\geq (H_m - K\|g'\|_1)|t - 2i|^{-1} \\ &\geq C(K_1 - \|g'\|_1)|t - 2i|^{-1}; \end{aligned}$$

where C is made independent of ϵ and λ . (2.4) follows immediately from (2.6). \square

Lemma 2.6. *If $F \in \mathbf{A}$, then $G(F, \bar{F})(t) = O(t^{-\tau})$, as $|t| \rightarrow \infty$, t in any angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$.*

Proof. From Remark 1.5, Remark 2.2 and Lemma 2.5, with \mathcal{D} being an angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$, the proof follows. \square

Remark 2.7. Let H_m, K, K_1 be as in (2.5) of Lemma 2.5 with \mathcal{D} being an angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$, then H_m, K, K_1 can be chosen to be independent of ϵ and λ since λ is in a fixed compact subset of $(0, 1)$. If $\|F'\|_{1,\mathcal{D}} < \frac{K_1}{2}$ then,

$$(2.7) \quad |\xi - 2i||F' + H| \geq (H_m - |\xi - 2i|^{-\tau}\|F'\|_1) \geq C(K_1 - \|F'\|_1) > C\frac{K_1}{2} > 0;$$

so $F' + H \neq 0$ holds in \mathcal{D} .

Remark 2.8. Let $\|F'\|_{1,\mathcal{D}} < \frac{K_1}{2}$, by Remark 2.7, we have $\bar{F}' + \bar{H} \neq 0$ in \mathcal{R}^* , so $G(F, \bar{F})$ is analytic in \mathcal{D} which is an angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$.

Definition 2.9. Let $F \in \mathbf{A}, \|F'\|_{1,\mathcal{D}} < \frac{K_1}{2}$, \mathcal{D} is an angular subdomain of $\mathcal{R} \cap \mathcal{R}^*$ and $(-\infty, \infty) \subset \mathcal{D}$, define operator I so that

$$(2.8) \quad I(F)[\xi] = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \bar{F})(t)dt}{t - \xi} \text{ for } \text{Im } \xi < 0.$$

Lemma 2.10. *Let $F \in \mathbf{A}$ be analytic in $\text{Im } \xi > 0$ as well. If $F(\xi)$ satisfies equation (1.14) on the real axis, then F satisfies for $\xi \in \{\text{Im } \xi < 0\}$*

$$(2.9) \quad F'(\xi) = \frac{1}{2\epsilon^4} \{ -[G_2(F, I)^2(\bar{F}' + \bar{H}) + 2\epsilon^4 G_1(\bar{F})] + (\bar{F}' + \bar{H})G_2(F, I)\sqrt{G_2(F, I)^2 - 4\epsilon^4} \}$$

where

$$(2.10) \quad G_1(\bar{F}) = -\bar{F}' + H - \bar{H};$$

$$(2.11) \quad G_2(F, I) = (F + \epsilon^2 I(G)).$$

Proof. Since F is analytic in $\{Im \xi > 0\}$ and satisfies equation (1.14), by using Poisson formula, we have for $Im \xi > 0$:

$$(2.12) \quad \begin{aligned} F(\xi) &= -\frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{(t-\xi)} \frac{1}{|F'(t) + H(t)|} \text{Im} [F'(t) + H(t)] \\ &= \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, \bar{F})(t) dt}{t-\xi} \end{aligned}$$

Using Plemelj Formula (see for eg. Carrier, Krook & Pearson, 1966), we analytically extend above equation to the lower half plane to obtain

$$(2.13) \quad F(\xi) = -\epsilon^2 I(\xi) + i\epsilon^2 \frac{(F'(\xi) + H(\xi)) - (\bar{F}'(\xi) + \bar{H}(\xi))}{(F'(\xi) + H(\xi))^{1/2} (\bar{F}'(\xi) + \bar{H}(\xi))^{1/2}}, \text{ for } Im \xi < 0;$$

Squaring the above equation to obtain

$$(2.14) \quad \epsilon^4 (F')^2 + G_3(F, \bar{F}, I) F' + G_4(F, \bar{F}, I) = 0$$

where

$$(2.15) \quad G_4(F, \bar{F}, I) = H(F + \epsilon^2 I(G))^2 (\bar{F}' + \bar{H}) + \epsilon^4 (G_1(\bar{F}))^2;$$

$$(2.16) \quad G_3(F, I, \bar{F}) = (\bar{F}' + \bar{H})(F + \epsilon^2 I(G))^2 + 2\epsilon^4 G_1(\bar{F});$$

Solving F' from above leading to (2.9). \square

Definition 2.11. Define two rays:

$$\begin{aligned} r_0^+ &= \{\xi : \xi = \rho e^{i(\varphi_0 + \frac{1}{2}\mu)}, 0 < \rho < \infty\}; \\ r_0^- &= \{\xi : \xi = \rho e^{i(\pi - \varphi_0 - \frac{1}{2}\mu)}, 0 < \rho < \infty\}; \end{aligned}$$

Let r_0 be the directed contour along $r_0^- \cup r_0^+$ from left to right (See Fig. 4).

Definition 2.12. If $F \in \mathbf{A}$, define $F'_-(\xi)$ for ξ below r_0 :

$$(2.17) \quad F'_-(\xi) = -\frac{1}{2\pi i} \int_{r_0} \frac{\bar{F}'(t)}{t-\xi} dt;$$

Remark 2.13. Since $r_0 \subset \mathcal{R}^*$ and \bar{F} satisfies (1.17) in \mathcal{R}^* , the above integral is well defined. It is obvious that $F'_-(\xi)$ is analytic below r_0 . Also, if $F \in \mathbf{A}_0^-$ only and the relation $[F(-t^*)]^* = F(t)$ were invoked to define F in \mathcal{R}^+ , then it is possible to use the symmetry between contours r_0^+ and r_0^- to write

$$(2.18) \quad \begin{aligned} F'_-(\xi) &= -\frac{1}{2\pi i} \int_{r_0} \frac{\bar{F}'(t) - \bar{F}'(0)}{t-\xi} dt + \bar{F}'(0) \\ &= -\frac{1}{2\pi i} \left[\int_{r_0^-} \frac{(\bar{F}'(t) - \bar{F}'(0)) dt}{t-\xi} + \int_{r_0^+} \frac{([\bar{F}'(-t^*)]^* - \bar{F}'(0)) dt}{t-\xi} \right] + \bar{F}'(0) \\ &= -\frac{1}{2\pi i} \int_{r_0^-} \left[\frac{(\bar{F}'(t) - \bar{F}'(0)) dt}{t-\xi} - \frac{([\bar{F}'(t)]^* - [\bar{F}'(0)]^*)(dt)^*}{t^* + \xi} \right] + \bar{F}'(0) \end{aligned}$$

This alternate expression is equivalent to (2.15) when symmetry condition $Im F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$ holds; however, (2.18) defines an analytic function F_- below r_0 even when symmetry condition is relaxed, as it is for the half Problem later in §2. We also notice that F_- , as defined by (2.18) satisfies symmetry condition even when $F \in \mathbf{A}^-$ does not.

Lemma 2.14. *If $F \in \mathbf{A}$ and F is also analytic in $\text{Im } \xi > 0$, then $\bar{F}'(\xi) = F'_-(\xi)$, for $\xi \in \mathcal{R}$.*

Proof. Since F is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$, then \bar{F} is analytic in $\mathcal{R}^* \cup \{\text{Im } \xi < 0\}$. We use property (1.17) and the Cauchy Integral formula to obtain

$$F'_-(\xi) = -\frac{1}{2\pi i} \int_{r_0} \frac{\bar{F}'(t)}{t - \xi} dt = \bar{F}'(\xi), \text{ for } \xi \in \mathcal{R};$$

□

Lemma 2.15. *Let $\Gamma = \{t, t = \rho e^{i\varphi}, 0 \leq \rho < N\}$ be a ray segment, \mathcal{D} is some connected set such that $\Gamma \cap \mathcal{D} = \{0\}$ and*

$$(2.19) \quad |\xi - t| \geq m|\rho e^{i\varphi_1} - |\xi||; \text{ for } \xi \in \mathcal{D}, t = \rho e^{i\varphi} \in \Gamma;$$

for some constants $m > 0$ and φ_1 independent of ϵ . Assume g to be a continuous function on Γ with $|g(t) - g(0)| \leq K\sqrt{|t|}$ for $t \in \Gamma \cap B_\nu$, ν is as in Remark 1.6. Then

$$(2.20) \quad \sup_{\mathcal{D} \cap B_\nu} \left| \int_\Gamma \frac{g(t) - g(0)}{(t - \xi)} dt - \int_\Gamma \frac{g(t) - g(0)}{t} dt \right| \leq C \left((\sqrt{|\nu|})^{-1} \sup_{\Gamma/B_\nu} |g| + K \right) \sqrt{|\xi|};$$

and

$$(2.21) \quad \sup_{\mathcal{D} \cap B_\nu} \left| \int_\Gamma \frac{g(t) - g(0)}{t} dt \right| \leq C \left(|\log \nu| \sup_{\Gamma/B_\nu} |g| + K\sqrt{|\nu|} \right);$$

where constant C depends on φ_1 , and m only.

Proof. On Γ , $t = \rho e^{i\varphi}$, $|dt| = d\rho$. Breaking up the integral in (2.20) into two parts:

$$(2.22) \quad \begin{aligned} \int_\Gamma \frac{g(t) - g(0)}{(t - \xi)} dt - \int_\Gamma \frac{g(t) - g(0)}{t} dt &= \xi \int_{\Gamma \cap B_\nu} \frac{g(t) - g(0)}{t(t - \xi)} dt \\ &\quad + \xi \int_{\Gamma/B_\nu} \frac{g(t) - g(0)}{t(t - \xi)} dt \end{aligned}$$

For the first integral in (2.22), we use (2.19) to obtain (on scaling ρ by $|\xi|$):

$$\left| \xi \int_{\Gamma \cap B_\nu} \frac{g(t) - g(0)}{t(t - \xi)} dt \right| \leq \frac{K}{m} \sqrt{|\xi|} \int_0^1 \frac{s^{-1/2} ds}{[(s \cos \varphi_1 - 1)^2 + s^2 \sin^2 \varphi_1]^{1/2}}$$

For the second integral we use (2.19) to obtain

$$\begin{aligned} \left| \xi \int_{\Gamma/B_\nu} \frac{g(t) - g(0)}{t(t - \xi)} dt \right| &\leq |\xi| \frac{\sup_{\Gamma/B_\nu} |g|}{m} \int_{|\nu|}^N \frac{ds}{s[(s \cos \varphi_1 - |\xi|)^2 + s^2 \sin^2 \varphi_1]^{1/2}} \\ &\leq C(\sqrt{|\nu|})^{-1} \left(\sup_{\Gamma/B_\nu} |g| \right) \sqrt{|\xi|} \end{aligned}$$

(2.21) can be proved similarly. □

Remark 2.16. Note if Γ is in \mathcal{D}'_c , an angular subset of \mathcal{D}_c (complement of \mathcal{D}), then (2.19) hold.

Definition 2.17.

$$(2.23) \quad \Omega_0 = \left\{ \xi : \xi \text{ is below } \{ \xi = \rho e^{i(\pi - \varphi_0 - \frac{1}{3}\mu)} \} \right. \\ \left. \cup \{ \xi = \rho e^{i(\varphi_0 + \frac{1}{3}\mu)} \} \right\}.$$

Remark 2.18. $\mathcal{R} \cup \{\text{Im } \xi < 0\}$ is an angular subset of Ω_0 , and Ω_0 is itself an angular subset of the region $\{\xi \text{ below } r_0\}$.

Lemma 2.19. *Let $F \in \mathbf{A}_0^-$, then*

$$\sup_{\xi \in \Omega_0/B_\nu} |(\xi - 2i)^{k+\tau} F_-^{(k)}(\xi)| \leq K_2 \frac{\sup_{\xi \in \mathcal{R}/B_\nu} |(\xi - 2i)^\tau F|}{|\nu|^k}, \quad k = 0, 1, 2;$$

and

$$\sup_{\xi \in \Omega_0 \cap B_\nu} \left| \frac{F'_-(\xi) - F'_-(0)}{\sqrt{\xi}} \right| \leq K_2 \left(\sup_{\xi \in \mathcal{R}^- \cap B_\nu} \left| \frac{F'_-(\xi) - F'_-(0)}{\sqrt{\xi}} \right| + |\nu|^{-3/2} \sup_{\xi \in \mathcal{R}/B_\nu} |F| \right);$$

where constant K_2 depends only on φ_0 .

Proof. From remarks 2.16-2.18, conditions (2.16)-(2.18) hold with $\Gamma = r_0$ and $\mathcal{D} = \Omega_0$. Using $\|\bar{F}\|_{0,r_0^-} \leq \|F\|_0^-$, (2.18) and applying Lemma 2.15, with $g = \bar{F}'$, we obtain the proof. \square

Remark 2.20. From now on, we choose $F \in \mathbf{A}_{0,\delta}^-$, with additional restriction,

$$(2.24) \quad \delta < \frac{K_3}{2} \equiv \frac{H_m}{2KK_2};$$

where H_m, K are as in (2.5) with $\mathcal{D} = \mathcal{R}$, while K_2 is defined as in Lemma 2.19. This ensures

$$(2.25) \quad |\xi - 2i| \|F'_- + \bar{H}\| \geq (H_m - K \|F'_-\|_{1,\mathcal{D}}) \geq (H_m - KK_2 \|F\|_0) \geq C \frac{K_3}{2} > 0;$$

so $F'_- + \bar{H} \neq 0$ in \mathcal{R} . On the other hand, from Cauchy integral formula and Lemma 2.12 in [22], we have $\|F'\|_{1,\mathcal{D}} \leq C \|F\|_0 \leq \frac{K_1}{2}$. Therefore from Remark 2.8, $G(F, F_-)$ is analytic in \mathcal{D}' which is subdomain of \mathcal{R} and contains real ξ axis.

Remark 2.21. Using Lemmas 2.5 and 2.19, $G(F, F_-)(t) = O(t^{-\tau})$ as $|t| \rightarrow \infty, t$ in any angular subset of $\mathcal{R} \cap \Omega_0$ including the real axis.

Definition 2.22. If $F \in \mathbf{A}_{0,\delta}$, define operator $I_1(F)$ so that for $\text{Im } \xi < 0$:

$$(2.26) \quad I_1(F)[\xi] = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, F_-)(t)}{t - \xi} dt,$$

Lemma 2.23. *If $F \in \mathbf{A}_{0,\delta}$ and F is analytic in $\text{Im } \xi > 0$ and also satisfies (1.14), then in the region $\{\text{Im } \xi < 0\} \cap \mathcal{R}$, F satisfies*

$$(2.27) \quad F'(\xi) = \frac{1}{2\epsilon^4} \left\{ -[G_2(F, I_1)^2(F_-)' + \bar{H}] + 2\epsilon^4 G_1(F'_-) \right. \\ \left. + (F'_- + \bar{H}) G_2(F, I_1) \sqrt{G_2(F, I_1)^2 - 4\epsilon^4} \right\};$$

Proof. Since conditions of Lemma 2.10 are satisfied, (2.9) holds. Since F is analytic in $\mathcal{R} \cup \{\text{Im } \xi > 0\}$ with property (1.17), from Lemma 2.14, $F_- = \bar{F}$; hence $I_1(F)(\xi) = I(F)(\xi)$. Therefore (2.9) implies (2.27). \square

Lemma 2.24. *If $F \in \mathbf{A}_{0,\delta}$, and F satisfies equation (2.27) in $\mathcal{R} \cap \{Im \xi < 0\}$, then F is analytic in $\mathcal{R} \cup \{Im \xi > 0\}$ and satisfies (1.14) on the real ξ axis.*

Proof. Note on using expression from (2.26), (2.27) can be rewritten as:

$$(2.28) \quad F(\xi) = -\epsilon^2 I_1(F)(\xi) + i\epsilon^2 G(F, F_-)[\xi], \text{ for } Im \xi < 0;$$

where operator G is defined by (2.3). Analytically continuing the above equation to the upper half plane, we have:

$$(2.29) \quad F(\xi) = \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, F_-)(t)dt}{t - \xi}, \text{ for } Im \xi > 0;$$

so $F(\xi)$ is analytic in the upper half plane. From Lemma 2.14, $F_-(\xi) = \bar{F}(\xi)$; hence on the real ξ -axis, $F_-(\xi) = F^*(\xi)$. Equation (2.29) reduces to:

$$F(\xi) = -\frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{t - \xi} \frac{1}{|F'(t) + H(t)|^{1/2}} Im [F'(t) + H(t)], \text{ for } Im \xi > 0;$$

On taking the limit $Im \xi \rightarrow 0^+$, the above implies (1.14). \square

Because of Remark 1.10 about equivalence of condition (iii) to $Im F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$, Lemmas 2.23 and 2.24 imply:

Theorem 2.25. *The finger problem is equivalent to*

Problem 1: *Find function $F \in \mathbf{A}_{0,\delta}$, satisfying $Im F = 0$ on $\{Re \xi = 0\} \cup \mathcal{R}$, so that (2.27) is satisfied in $\mathcal{R} \cap \{Im \xi < 0\}$.*

2.2. Formulation of Problem 2. Let $\alpha_1 > 0$ be a fixed constant independent of ϵ so that $-\alpha_1 i \in \mathcal{R}$. Define two rays (see Figure 4):

Definition 2.26.

$$r_1^+ = \{\xi : \xi = -\alpha_1 i + \rho e^{-i\varphi_0}, 0 < \rho < \infty\};$$

$$r_1^- = \{\xi : \xi = -\alpha_1 i + \rho e^{i(\pi+\varphi_0)}, 0 < \rho < \infty\};$$

$r_1 = r_1^- \cup r_1^+$ is a directed contour from left to right (see Fig.4).

Definition 2.27. Let $F \in \mathbf{A}_{0,\delta}$. For ξ above r_1 , define

$$(2.30) \quad F_+(\xi) = \frac{1}{2\pi i} \int_{r_1} \frac{F(t)dt}{t - \xi}$$

Remark 2.28. F_+ as defined above is analytic for ξ above r_1 . Also, it is to be noted that for $F \in \mathbf{A}_0^-$ only, when $F(\xi) = [F(-\xi^*)]^*$ is invoked to define F on \mathcal{R}^+ , we can write

$$(2.31) \quad \begin{aligned} F_+(\xi) &= \frac{1}{2\pi i} \left[\int_{r_1^-} \frac{F(t)dt}{t - \xi} + \int_{r_1^+} \frac{[F(-t^*)]^* dt}{t - \xi} \right] \\ &= \frac{1}{2\pi i} \int_{r_1^-} \left[\frac{F(t)dt}{t - \xi} - \frac{[F(t)]^* dt^*}{t^* + \xi} \right] \end{aligned}$$

This expression is equivalent to (2.30) when symmetry condition: $Im F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$ is satisfied. However, even without symmetry, (2.31) still defines an analytic function for ξ above r_1 , with possible singularity at $\xi = -i\alpha_1$. It is easy to see that $F_+(\xi)$ satisfies symmetry condition on $\{Re \xi = 0\} \cup \mathcal{R} \cup \{Im \xi > -\alpha_1\}$ even when F does not. Further, if $F \in \mathbf{A}_0$ is also analytic in $\{Im \xi > 0\}$ then it is clear from (2.30) on closing the contour from the top that $F_+(\xi) = F(\xi)$.

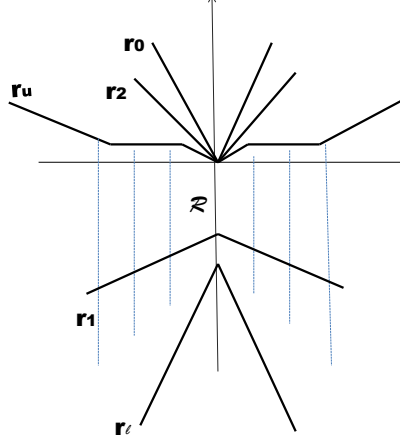


FIGURE 4. Rays defined in Subsection 2.2.

Definition 2.29.

$$(2.32) \quad \Omega_1 = \left\{ \xi : \xi \text{ is above } \left\{ \xi = -\frac{\alpha_1}{2}i + \rho e^{i(\pi + \frac{\varphi_0}{2})} \right\} \cup \left\{ \xi = -\frac{\alpha_1}{2}i + \rho e^{-i\frac{\varphi_0}{2}} \right\} \right\};$$

Remark 2.30. Ω_1 is an angular subset of the region $\{\xi : \xi \text{ above } r_1\}$, r_1 as in Definition 2.26.

Lemma 2.31. *Let $F \in \mathbf{A}_{0,\delta}^-$, $F' \in \mathbf{A}_{1,\delta_1}^-$, then*

$$\sup_{\xi \in \Omega_1} \left| (\xi - 2i)^{k+\tau} F_+^{(k)}(\xi) \right| \leq K_2 \|F\|_0^- \text{ for } k = 0, 1, 2.$$

where $K_2 > 0$ is independent of ϵ, λ .

Proof. The proof is very similar to that of Lemma 2.19. From remarks 2.16-2.30, conditions (2.16)-(2.18) hold with $\Gamma = r_1$ and $\mathcal{D} = \Omega_1$. Using $\|\bar{F}\|_{0,r_0^-} \leq \|F\|_0^-$, (2.31) and applying Lemma 2.15, with $g = F$, we obtain the proof. \square

Definition 2.32. We define two rays (see Figure 4):

$$r_2^+ = \{\xi : \xi = \rho e^{i\varphi_0 + \frac{1}{4}\mu}, 0 < \rho < \infty\};$$

$$r_2^- = \{\xi : \xi = \rho e^{i(\pi - \varphi_0 - \frac{1}{4}\mu)}, 0 < \rho < \infty\};$$

r_2 is a directed contour from left to right on the path $r_2^- \cup r_2^+$.

Remark 2.33. r_2 is an angular subset of $\Omega_0 \cap \Omega_1$ and \mathcal{R} is below r_2 .

Definition 2.34. If $F \in \mathbf{A}_{0,\delta}^-$, let F_-' be given by (2.18) and F_+ as in (2.30). Define operator I_2 so that

$$(2.33) \quad I_2(F)[\xi] = -\frac{1}{2\pi} \int_{r_2} \frac{G(F_+, F_-')(t)dt}{t - \xi}, \text{ for } \xi \text{ below } r_2;$$

Remark 2.35. Because of Lemmas 2.5, 2.19 and 2.31, $G(F_+, F_-)(t) \sim O(t^{-\tau})$ as $|t| \rightarrow \infty$ and analytic for t in any angular subset of $\Omega_0 \cap \Omega_1$ that includes r_2 ; so $I_2(F)[\xi]$ is analytic below r_2 . Also, from the symmetry of each of F_+ and F_- , it is not difficult to see that $I_2(F)[\xi]$ also satisfies symmetry condition on $\{Re \xi = 0\} \cap \mathcal{R}$.

Lemma 2.36. *Let $F \in \mathbf{A}_{0,\delta}^-$. Then $I_2(F) \in \mathbf{A}$, and $\|I_2(F)\|_0 \leq K_5$; where $K_5 > 0$ is independent of ϵ and λ .*

Proof. From Lemma 2.19 and Lemma 2.31,

$$\|F_-^{(k)}\|_{k,r_2} \leq K_2 \|F\|_0^-, \quad \|F_+^{(k)}\|_{k,r_2} \leq K_2 \|F\|_0^-$$

Applying Lemma 2.5 (with $\mathcal{D} = r_2$) and Lemma 2.15 (with $\Gamma = r_2$ and $\mathcal{D} = \mathcal{R}$) to (2.33), we complete the proof. \square

Lemma 2.37. *If $F \in \mathbf{A}_{0,\delta}$, and F satisfies (2.27) in $\mathcal{R} \cap \{Im \xi < 0\}$, then for $\xi \in \mathcal{R}$, F satisfies*

$$(2.34) \quad F'(\xi) = \frac{1}{2\epsilon^4} \{ -[G_2(F, I_2)^2 (F^-)' + \bar{H}] + 2\epsilon^4 G_1(F^-) + ((F^-)' + \bar{H}) \sqrt{G_2(F, I_2)^2 - 4\epsilon^4} \};$$

Proof. If δ is small enough, from Lemmas 2.19 and 2.31, $|(\xi - 2i)^{1+\tau} F_-'|$ and $|(\xi - 2i)^{1+\tau} F_+'|$ and are each small in the domain $\Omega_0 \cap \Omega_1$ which contains the region between r_2 and r_1 ; hence $F_-' + \bar{H} \neq 0$ and $F_+' + H \neq 0$ in that domain. From Lemma 2.24, F is analytic in $\{Im \xi > 0\}$; hence $F_+ = F$. By deforming the contour r_2 in (2.33) back to real axis, it follows $I_2(F)(\xi) = I_1(F)(\xi)$, for $Im \xi < 0$; By analytic continuation, F satisfies (2.34) for $\xi \in \mathcal{R}$. \square

Definition 2.38. Problem 2: Find function $F \in \mathbf{A}_{0,\delta}^-$ so that F satisfies symmetry condition $Im F = 0$ on $\mathcal{R} \cap \{Re \xi = 0\}$ and equation (2.34) in \mathcal{R}^- .

Theorem 2.39. *Let $F \in \mathbf{A}_{0,\delta}^-$. If $F(\xi)$ satisfies the symmetry condition $Im F = 0$ on $\mathcal{R} \cap \{Re \xi = 0\}$ and the equation (2.34) in \mathcal{R}^- , then for sufficiently small ϵ and δ , F is a solution to **Problem 1** (and hence a solution to the original Finger Problem).*

The proof of Theorem 2.39 will be given later after several lemmas.

Lemma 2.40. *Assume $F \in \mathbf{A}_{0,\delta}^-$ and F satisfies integral equation (2.34) in \mathcal{R}^- as well as the symmetry condition $Im F(\xi) = 0$ for $\xi \in \{Re \xi = 0\} \cap \mathcal{R}$. Let $U(\xi) = F(\xi) - F_+(\xi)$, then*

- (1) $U(\xi)$ is analytic in $\mathcal{R} \cap \{Im \xi > 0\}$.
- (2) For $\xi \in \mathcal{R}^- \cap \{Im \xi > 0\}$, U satisfies:

$$(2.35) \quad \epsilon^2 U'(\xi) + \tilde{Q}(\xi) U(\xi) = \bar{M}(U) + M(U);$$

where

$$(2.36) \quad \tilde{Q}(\xi) = \frac{2iH^{3/2}\bar{H}^{1/2}}{H + \bar{H}} = i \frac{(\xi + i\gamma)^{3/2}(\xi - i\gamma)^{1/2}}{\xi(\xi^2 + 1)},$$

and $\bar{M}(\xi) = \bar{M}(U)[\xi] \equiv [-iR_1(\xi) + \tilde{Q}(\xi)]U$; where

$$(2.37) \quad M(U)(\xi) := -\frac{\epsilon^2 i}{2\pi} R_1(\xi) \int_{-\infty}^{\infty} \frac{\mathcal{R}_1^{-1}(t) U'(t)}{t - \xi} dt, \text{ for } Im \xi > 0;$$

and $\mathcal{R}_1(\xi) = R_1(F, F_-, F_+)[\xi]$, where operator R_1 is defined by:

$$(2.38) \quad R_1(F, F_-, F_+) = \frac{[(F'_+ + H)^{1/2} + (F' + H)^{1/2}](F'_+ + H)^{1/2}(F' + H)^{1/2}(F'_- + \bar{H})^{1/2}}{[(F'_- + H) + (F' + H)^{1/2}(F'_+ + H)^{1/2}]}.$$

Proof. Since each of F and F_+ are analytic in $\mathcal{R} \cap \{Im \xi > 0\}$, it follows that $U = F - F_+$ is also analytic in $\mathcal{R} \cap \{Im \xi > 0\}$; hence statement (1) follows. Since F satisfies (2.35) in \mathcal{R}^- , symmetry condition and Schwarz reflection principle that relates F and its derivatives in \mathcal{R}^+ to their values in \mathcal{R}^- guarantees that (2.35) is satisfied in \mathcal{R} . But equation (2.35) can be rewritten as:

$$(2.39) \quad F(\xi) = -\epsilon^2 I_2(\xi) + i\epsilon^2 G(F, F_-)(\xi),$$

Then, on deforming the contour for I_2 from r_2 to $(-\infty, \infty)$,

$$(2.40) \quad F(\xi) = -\epsilon^2 I_3(\xi) - i\epsilon^2 [G(F_+, F_-)(\xi) - G(F, F_-)(\xi)], \text{ for } \xi \text{ above } (-\infty, \infty)$$

where

$$(2.41) \quad I_3(\xi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(F_+, F_-)(t)}{t - \xi} dt, \text{ for } \xi \text{ above } r_1;$$

It is clear that $I_3(\xi)$ is analytic above r_1 ; indeed from contour deformation of (2.30) and (2.41) and analyticity and decay properties of $G(F_+, F_-)$ on \mathcal{R} itself, it is clear that $I_3 \in \mathbf{A}_0$ and analytic in $\mathcal{R} \cup \{Im \xi > 0\}$. Substituting for F from (2.41) into (2.34), it follows that for $Im \xi > 0$,

$$(2.42) \quad \begin{aligned} F_+(\xi) &= -\epsilon^2 I_3(\xi) - \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{[G(F_+, F_-)(t) - G(F, F_-)(t)]}{t - \xi} dt \\ &= \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F, F_-)(t)}{t - \xi} dt \text{ for } Im \xi > 0; \end{aligned}$$

Subtracting (2.42) from (2.40), we obtain for $Im \xi > 0$:

$$(2.43) \quad \begin{aligned} U(\xi) &= -i\epsilon^2 [G(F_+, F_-)(\xi) - G(F, F_-)(\xi)] \\ &\quad + \frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{G(F_+, F_-)(t) - G(F, F_-)(t)}{t - \xi} dt \end{aligned}$$

Using (2.3), and the definition of G_1 and G_4 in (2.41) and (2.38),

$$(2.44) \quad G(F, F_-) - G(F_+, F_-) = R_1^{-1}(F, F_-)U';$$

From (2.44), (2.43) $U(\xi)$ satisfies the equation (2.35) for $Im \xi > 0$: □

Definition 2.41. We define two rays r_3^- and r_3^+ , i.e.

$$\begin{aligned} r_3^- &= \{\xi : \xi = re^{i(\pi - \phi_0/2)}, r > 0\} \\ r_3^+ &= \{\xi : \xi = re^{i(\phi_0/2)}, r > 0\} \end{aligned}$$

and Ω_3 is defined to be the region between r_3 and r_u .

Definition 2.42.

$$(2.45) \quad g_1(\xi) = \exp\left\{-\frac{P(\xi)}{\epsilon^2}\right\},$$

$$(2.46) \quad g_2(\xi) = \exp\left\{\frac{P(\xi)}{\epsilon^2}\right\},$$

where

$$(2.47) \quad P(\xi) = \int_{-\infty}^{\xi} \tilde{Q}(t) dt = \int i \frac{(\xi + i\gamma)^{3/2} (\xi - i\gamma)^{1/2}}{\xi(\xi^2 + 1)} d\xi;$$

Lemma 2.43. *Let U be as in Lemma 2.40, then U satisfies*

$$(2.48) \quad U = \frac{1}{\epsilon^2} g_1(\xi) \int_{-\infty}^{\xi} (\bar{M}(t) + M(t)) g_2(t) dt.$$

Proof. Since U satisfies (2.35), we have

$$U(\xi) = Ch_1(\xi) + \frac{1}{\epsilon^2} g_1(\xi) \int_{-\infty}^{\xi} (\bar{M}(t) + M(t)) g_2(t) dt$$

C has to be zero since $U(\infty) = 0$, $h_1(\infty) \neq 0$ and Lemma. \square

Remark 2.44. We will show that $U = 0$. Since F , F^+ and hence U is known to be analytic in \mathcal{R} and continuous upto its boundary, it is enough to show that $U = 0$ on Ω_3^- . We will do so by showing that equation (2.48) forms a contraction map in the space of functions U on \mathcal{R}^- , with norm

$$\|U\|_{\Omega_3} = \sup_{\xi \in \Omega_3} |\xi - 2i|^\tau |U(\xi)| + \epsilon^2 \sup_{B_\nu \cap \Omega_3} \left| \frac{U'(t) - U'(0)}{\sqrt{t}} \right| + |U'(0)|$$

It is to be noted that the integration in (2.48) can be performed on the path $\mathcal{P}(\xi, \infty)$ contained in Ω_3^- , so that on this path on any \mathbf{C}^1 segment, $\frac{d}{ds} \operatorname{Re} P(t(s)) < 0$ for arc length s increasing in the direction of ∞ , Lemmas 2.53-2.56 are valid and hence the integral operator in (2.48) is bounded when restricted to functions on Ω_3^- . Further note that since U are analytic in $\mathcal{R} \cup \{Im \xi < 0\}$ and satisfies symmetry condition, it follows that U on Ω_3^- completely determines U and its derivatives for ξ real. This is crucial in controlling \mathcal{M} on r_3^- , as necessary.

Lemma 2.45. *Let $F \in \mathbf{A}_{0,\delta}$, then for $\xi \in \mathcal{R}$*

$$(2.49) \quad |R_1(F, F_-)(\xi)| \leq C/|\xi - 2i|; \quad |1/R_1(F, F_-)(\xi)| \leq C|\xi - 2i|;$$

where C is independent of ϵ and λ .

Proof. The lemma follows from (2.38) and Lemma 2.40. \square

Lemma 2.46. *let $U(\xi)$ be as in Lemma 2.40, then*

$$(2.50) \quad \sup_{\xi \in \mathcal{D}} |\xi - 2i|^{1+\tau} |U'(\xi)| \leq \frac{C}{|\nu|} \|U\|_{0, r_3^-};$$

when $\mathcal{D} = (-\infty, -\nu)$.

Proof. Since U is analytic in the region under r_3 and continuous upto its boundary, by Cauchy integral formula:

$$U'(\xi) = -\frac{1}{2\pi i} \int_{r_3} \frac{U(t)}{(t - \xi)^2} dt; \text{ for } \xi \text{ in } \mathcal{D}$$

Applying Lemma 2.11 in [22], with Γ chosen to be r_u , we have $|\xi - 2i|^{1+\tau} |U'(\xi)| \leq C \|U\|_{0, r_3^-}$ for $\xi \in (-\infty, -3)$.

For $\xi \in (-\infty, -3)$, we split the integral into

$$\begin{aligned} U'(\xi) &= -\frac{1}{2\pi i} \left(\int_{r_3 \cap B_3} + \int_{r_3/B_3} \right) \frac{U(t)}{(t-\xi)^2} dt \\ &=: U_1 + U_2; \end{aligned}$$

Applying Lemma 2.11 in [22], with Γ chosen to be r_u/B_3 , we have $|U_2|(\xi) \leq C\|U\|_{0,r_3^-}$ for $\xi \in (-3, -\nu)$.

For $t = re^\theta \in r_3 \cap B_3$ and $\xi \in (-3, -\nu)$, using $|t - \xi|^2 = (r \cos \theta - \xi)^2 + r^2 \sin^2 \theta$, we have

$$\begin{aligned} |U_1(\xi)| &= \left| -\frac{1}{2\pi i} \int_{r_3 \cap B_3} \frac{U(t)}{(t-\xi)^2} dt \right| \\ &\leq C \left| \int_0^3 \frac{\frac{\sup|U(t)|}{r_3}}{(r \cos \theta - \xi)^2 + r^2 \sin^2 \theta} dr \right| \leq C \frac{\sup|U(t)|}{|\xi|} \end{aligned}$$

Hence the lemma follows. \square

Lemma 2.47. *let $U(\xi)$ be as in Lemma 2.40, then*

$$(2.51) \quad \sup_{\xi \in (-\nu, 0)} |U'(\xi) - U'(0)| \leq \frac{C\sqrt{|\xi|}}{|\nu|^{3/2}} \|U\|_{\Omega_3}.$$

Proof.

$$U'(\xi) = -\frac{1}{2\pi i} \int_{r_4} \frac{U'(t)}{(t-\xi)} dt; \text{ for } \xi \text{ in } (-\nu, 0)$$

where r_4 is a line between r_3 and r_u defined by

$$\begin{aligned} r_4^- &= \{\xi : \xi = re^{i(\pi-2\phi_0/3)}, r > 0\} \\ r_4^+ &= \{\xi : \xi = re^{i(2\phi_0/3)}, r > 0\}. \end{aligned}$$

Applying Lemma 2.15, we obtain

$$|U'(\xi) - U'(0)| \leq C\sqrt{|\xi|} \left(\frac{\sup_{r_4/B_\nu} |U'(t)|}{\sqrt{|\nu|}} + \|U\|_{0,r_4} \right).$$

From Cauchy integral formula, for $t \in r_4/B_\nu$ we have

$$U'(t) = -\frac{1}{2\pi i} \int_{\partial\Omega_3} \frac{U(s)}{(s-t)^2} ds;$$

since $|t - s| \geq C|s|e^{i\theta} - |t|$, for $t \in r_4/B_\nu$ and $s \in \partial\Omega_3$ we have

$$\sup_{r_4/B_\nu} |U'(t)| \leq C \frac{\sup_{\partial\Omega_3} |U(s)|}{|\nu|};$$

hence the lemma follows. \square

Lemma 2.48. *let $U(\xi)$ be as in Lemma 2.40, then $\sup_{(-\nu, 0)} |U'(\xi)| \leq \frac{C}{|\nu|} \|U\|_{\Omega_3}$.*

Proof. The lemma follows from Cauchy integral formula and the above lemma. \square

Lemma 2.49. *Let $F \in \mathbf{A}_{0,\delta}$. Let $\bar{M}(U)$ be as defined in (2.36), then*

$$(2.52) \quad |\bar{M}(U)(\xi)| \leq C\delta_2(1 + |\xi|)^{-1-2\tau}\|U\|_{\Omega_3};$$

C is some constant independent of ϵ and

$$(2.53) \quad \delta_2 = \left\{ (\|F\|_0 + 1)^{1/2} - 1 \right\}.$$

Proof. The lemma follows from

$$(2.54) \quad |(-iR_1 + \tilde{Q})| \leq C|t - 2i|^{-1-\tau}\delta_2, \text{ for } t \in \Omega_3.$$

□

Lemma 2.50. *Let $F \in \mathbf{A}_{0,\delta}$. Let $M(U)$ be as defined in (2.37), then for $\xi \in \Omega/B_\nu$*

$$(2.55) \quad |M(U)(\xi)| \leq C \frac{\epsilon^2}{|\nu|^2} (1 + |\xi|)^{-1-\tau}\|U\|_{\Omega_3};$$

and

$$(2.56) \quad \left| \frac{d}{d\xi} M(U)(\xi) \right| \leq C \frac{\epsilon^2}{|\nu|^2} (1 + |\xi|)^{-2-\tau}\|U\|_{\Omega_3};$$

C is some constant independent of ϵ .

Proof. Using Lemma 2.45 and Lemma 2.46:

$$(2.57) \quad |R_1^{-1}U'(t)| \leq \frac{C}{|\nu|} |t - 2i|^{-\tau}\|U\|_0, \text{ for } t \in (-\infty, -\nu);$$

Applying Lemmas 2.15, and Lemmas 2.45-2.46 to (2.37), we have the lemma.

□

Lemma 2.51. *Let $F \in \mathbf{A}_{0,\delta}$. Let $M(U)$ be as defined in (2.36), then for $\xi \in \Omega_3 \cap B_\nu$, then*

$$(2.58) \quad \sup_{\xi \in (-\nu, 0)} |M'(\xi) - M'(0)| \leq \frac{C\sqrt{|\xi|}}{|\nu|^2} \|U\|_{\Omega_3}.$$

Proof. The Lemma follows from Lemmas 2.46-2.48.

□

Definition 2.52. Let \mathcal{Q} be any connected set in complex ξ -plane, we introduce norms: $\|F(\xi)\|_{j,\mathcal{Q}} := \sup_{\xi \in \mathcal{Q}} |(\xi - 2i)^{j+\tau} F(\xi)|$, $j = 0, 1, 2$; and $\mathbf{C}(\mathcal{Q})$ is the function space of all continuous functions on \mathcal{Q} .

Lemma 2.53. *If $N \in \mathbf{C}(\mathcal{Q})$, $N' \in \mathbf{C}(\mathcal{Q})$ where \mathcal{Q} is a path $\mathcal{P}(\xi, \infty)$ on which $\text{Re}(P(t) - P(\xi))$ decreases monotonically as $\text{Re } t$ increases, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{\infty}^{\xi} N(t) g_2(t) dt \in \mathbf{C}_0, \text{ and } \|f_1\|_0 \leq K_1 (\|N\|_1 + \|N'\|_2);$$

Proof. By integration by parts

$$\begin{aligned}
|f_1(\xi)| &= \left| \frac{1}{\epsilon^2} \int_{\mathcal{P}(\xi, \infty)} N(t) \exp\left\{-\frac{1}{\epsilon^2}(P(\xi) - P(t))\right\} dt \right| \\
&= \left| \frac{1}{\epsilon^2} \int_{\mathcal{P}(\xi, \infty)} \frac{N(t)}{-\frac{d}{ds}P(t)} d\left(\exp\left\{-\frac{1}{\epsilon^2}(P(\xi) - P(t))\right\}\right) \right| \\
&\leq \|N\|_1 \times \int_0^\infty \frac{|(t-2i)^{-1-\tau}| \left|\frac{d^2}{dt^2}P(t)\right|}{\left|\frac{d}{dt}P(t)\right|^2} \left[\exp\left\{-\frac{1}{\epsilon^2}(ReP(\xi) - ReP(t))\right\}\right] dt \\
&\quad + \|N'\|_2 \times \int_0^\infty \frac{|(t-2i)^{-2-\tau}|}{\left|\frac{d}{dt}P(t)\right|} \left[\exp\left\{-\frac{1}{\epsilon^2}(ReP(\xi) - ReP(t))\right\}\right] dt
\end{aligned}$$

Since

$$\begin{aligned}
\left|\frac{d}{dt}P(t)\right| &\geq C|t(s) - 2i|^{-1}; \\
\left|\frac{d^2}{dt^2}P(t)\right| &\leq C|t(s) - 2i|^{-2}; \\
\frac{|(t-2i)^{-2-\tau}|}{\left|\frac{d}{dt}P(t)\right|} &\leq C|\xi - 2i|^{1-\tau};
\end{aligned}$$

So $\|f_1\|_0 \leq K_1(\|N\|_1 + \|N'\|_2)$ and the lemma follows. \square

Lemma 2.54. *If $N \in \mathbf{C}(\mathcal{Q})$, where \mathcal{Q} is a path $\mathcal{P}(\xi, \infty)$ on which $Re(P(t) - P(\xi))$ decreases monotonically as $Re t$ increases, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{\mathcal{P}(\xi, \infty)} N(t) g_2(t) dt \in \mathbf{C}(\mathcal{Q}) \text{ and } \|f_1\|_0 \leq \frac{K_1}{\epsilon^2} \|N\|_1;$$

Proof.

$$\begin{aligned}
|f_1(\xi)| &= \left| \frac{1}{\epsilon^2} \int_{\mathcal{P}(\xi, \infty)} N(t) \exp\left\{-\frac{1}{\epsilon^2}(P(\xi) - P(t))\right\} dt \right| \\
&\leq \frac{1}{\epsilon^2} \|N\|_1 \int_{|\xi|}^\infty |(t-2i)^{-1-\tau}| dt \leq \frac{K_1}{\epsilon^2} \|N\|_1 |\xi - 2i|^{-\tau}.
\end{aligned}$$

\square

Lemma 2.55. *Let $R_\epsilon > R$ be a number which will be chosen later and be dependent on ϵ . If $N(t) \leq C|t|^{-\sigma}$ for $t \in \mathcal{P}(\xi, R_\epsilon^l)$, where $R_\epsilon^l = r_l^- \cap \{|\xi| = R_\epsilon\}$ and $\mathcal{P}(\xi, R_\epsilon^l) \subset \mathcal{R}^- \cap \{R \leq |\xi| \leq R_\epsilon\}$ is a path on which $Re(P(t) - P(\xi))$ decreases monotonically as $Re t$ increases, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{R_\epsilon^l}^\xi N(t) g_2(t) dt \in \mathbf{C}(\mathcal{P}(\xi, R_\epsilon^l)), \text{ and } \|f_1\|_0 \leq K_2 R_\epsilon^{(2-\sigma+\tau)} \sup(|t|^\sigma |N|);$$

Proof.

$$\begin{aligned}
(2.59) \quad |f_1(\xi)| &= \left| \frac{1}{\epsilon^2} \int_{\mathcal{P}(\xi, R_\epsilon^l)} N(t) \exp\left\{-\frac{1}{\epsilon^2}(P(\xi) - P(t))\right\} dt \right| \\
&\leq \sup(|t|^\sigma |N|) \times \int_0^1 \frac{|(t(s))|^{-\sigma}}{-\frac{d}{ds}ReP(t(s))} d\left[\exp\left\{-\frac{1}{\epsilon^2}(ReP(\xi) - ReP(t(s)))\right\}\right]
\end{aligned}$$

Since

$$-\frac{d}{ds} \operatorname{Re} P(t(s)) = -\operatorname{Re} (P'(t)t'(s)) \geq C|t(s)|^{-2};$$

$$\frac{|t(s)^{-\sigma}|}{-\frac{d}{ds} \operatorname{Re} P(t(s))} \leq C|t|^{2-\sigma};$$

So $\|f_1\|_0 \leq K_1 R_\epsilon^{2-\sigma+\tau} \sup |t^\sigma N(t)|$. \square

Lemma 2.56. *Let $N \in \mathbf{C}(\mathcal{P}(\xi, R^l))$, where $R^l = r_l^- \cap \{\xi : |\xi| = R\}$ and $\mathcal{P}(\xi, R^l) \subset \mathcal{R}^- \cap \{|\xi| \leq R\}$ is a path on which $\operatorname{Re} (P(t) - P(\xi))$ decreases monotonically as $\operatorname{Re} t$ increases, then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{R^l}^\xi N(t) g_2(t) dt \in \mathbf{C}(\mathcal{P}(\xi, R^l)), \text{ and } \|f_1\|_0 \leq K_1 \|N\|_1;$$

where K_1 is a constant independent of ϵ .

Proof.

$$(2.60) \quad |f_1(\xi)| = \left| \frac{1}{\epsilon^2} \int_{\mathcal{P}(\xi, R^l)} N(t) \exp\left\{-\frac{1}{\epsilon^2}(P(\xi) - P(t))\right\} dt \right| \\ \leq \sup N \times \int_0^1 \frac{1}{-\frac{d}{ds} \operatorname{Re} P(t(s))} d \left[\exp\left\{-\frac{1}{\epsilon^2}(\operatorname{Re} P(\xi) - \operatorname{Re} P(t(s)))\right\} \right]$$

Since for $-R \leq \operatorname{Re} t(s) \leq 0$, we have

$$-\frac{d}{ds} \operatorname{Re} P(t(s)) = -\operatorname{Re} (P'(t)t'(s)) \geq C;$$

$$\frac{1}{-\frac{d}{ds} \operatorname{Re} P(t(s))} \leq C;$$

So $\|f_1\|_0 \leq K_1 \sup |N(t)|$. \square

Lemma 2.57. *Let N be a continuous function on Ω_3^- and $\|N\|_{\Omega_3}$ be defined as in Remark 2.44. Let*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_\infty^\xi N(t) g_2(t) dt;$$

then

$$\epsilon^2 \sup_{\Omega_3^- \cap B_\nu} \left| \frac{f_1'(\xi) - f_1'(0)}{\sqrt{\xi}} \right| \leq K_1 \|N\|_{\Omega_3}$$

where K_1 is a constant independent of ϵ .

Proof.

$$f_1'(\xi) = \frac{1}{\epsilon^2} N(\xi) - \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_\infty^\xi N(t) g_2(t) dt;$$

$$f_1'(\xi) - f_1'(0) = \frac{1}{\epsilon^2} [N(\xi) - N(0)] - \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_\infty^\xi [N(t) - N(0)] g_2(t) dt;$$

we can break up the integral on the right hand side into

$$\begin{aligned}
& \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_{\infty}^{\xi} [N(t) - N(0)] g_2(t) dt \\
&= \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_{\infty}^{\nu_1} [N(t) - N(0)] g_2(t) dt + \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_{-\nu_1}^{\xi} [N(t) - N(0)] g_2(t) dt \\
&= \frac{P'(\xi)}{\epsilon^4} g_1(\xi) g_2(-\nu_1) \left(g_1(-\nu_1) \int_{\infty}^{-\nu_1} [N(t) - N(0)] g_2(t) dt \right) \\
&+ \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_{-\nu_1}^{\xi} [N(t) - N(0)] g_2(t) dt \\
&:= w_1(\xi) + w_2(\xi)
\end{aligned}$$

Using Lemma 2.56, we have

$$\left| g_1(-\nu_1) \int_{\infty}^{-\nu_1} [N(t) - N(0)] g_2(t) dt \right| \leq C \epsilon^2 \|N\|_{\Omega_3};$$

and using the fact that $P(t) \sim -\gamma^2 \log t$ for $|t| \leq \nu_1$, we obtain for $\xi \in B_\nu, \nu = O(\epsilon^{4/3})$,

$$\left| \frac{P'(\xi)}{\epsilon^4} g_1(\xi) g_2(-\nu_1) \right| \leq C |\xi|^{\frac{\gamma^2}{\epsilon^2}-1} |\nu_1|^{\frac{\gamma^2}{\epsilon^2}} \leq C \sqrt{|\xi|} \epsilon^{\frac{\gamma^2}{2\epsilon^2}-2};$$

hence $|w_1(\xi)| \leq C \sqrt{|\xi|} \|N\|_{\Omega_3} \epsilon^{\frac{\gamma^2}{2\epsilon^2}}$.

Using the fact that $P(t) \sim -\gamma^2 \log t$ and $g_1(\xi) \sim \xi^{\frac{\gamma^2}{\epsilon^2}}$ for $|\xi| \leq \nu_1$ and $|N(t) - N(0)| \leq \|N\|_{\Omega_3} |t|^{1/2}$ we obtain

$$\begin{aligned}
|w_2(\xi)| &= \left| \frac{P'(\xi)}{\epsilon^4} g_1(\xi) \int_{-\nu_1}^{\xi} N(t) g_2(t) dt \right| \\
&= \left| \frac{C}{\epsilon^4} \xi^{\frac{\gamma^2}{\epsilon^2}-1} \int_{-\nu_1}^{\xi} N(t) t^{\frac{\gamma^2}{\epsilon^2}} dt \right| \\
&\leq \frac{C}{\epsilon^2} \sqrt{|\xi|} \|N\|_{\Omega_3};
\end{aligned}$$

Therefore, the lemma follows. \square

Lemma 2.58. *Let $\mathcal{U}[\bar{M}] = \frac{1}{\epsilon^2} g_1(\xi) \int_{-\infty}^{\xi} \bar{M}(t) g_2(t) dt$, then $\|\mathcal{U}[U]\|_{\Omega_3} \leq C \delta_2 \|U\|_{\Omega_3}$.*

Proof. (1) For $\xi \in r_3 \cap \{|\xi| \geq R_\epsilon\}$, applying Lemma 2.54 and Lemma 2.49 with $N = \bar{M}$ and $\mathcal{P}(-\infty, \xi) = r_3 \cap \{|\xi| \geq R_\epsilon\}$, we have

$$(2.61) \quad |\mathcal{U}[\bar{M}]| \leq K_1 \frac{\delta_2}{\epsilon^2} R_\epsilon^{-\tau} |\xi - 2i|^{-\tau} \|U\|_{\Omega_3}.$$

(2) For $\xi \in r_3 \cap \{R \leq |\xi| \leq R_\epsilon\}$, applying Lemma 2.55 and Lemma 2.49 with $N = \bar{M}$ and $\mathcal{P}(R_\epsilon^l, \xi) = r_3 \cap \{R_\epsilon \geq |\xi| \geq R\}$, we have

$$(2.62) \quad |\mathcal{U}[\bar{M}]| \leq K_1 \left[\frac{\delta_2}{\epsilon^2} R_\epsilon^{-\tau} + R_\epsilon^{1-\tau} \delta_2 \right] |\xi - 2i|^{-\tau} \|U\|_{\Omega_3}.$$

(3) For $\xi \in r_3 \cap \{|\xi| \leq R\}$, applying Lemma 2.56 and Lemma 2.49 with $N = \bar{M}$ and $\mathcal{P}(R^l, \xi) = r_3 \cap \{R \geq |\xi| \geq 0\}$, we have

(2.63)

$$|\mathcal{U}[\bar{M}]| \leq K_1 \left[\frac{\delta_2}{\epsilon^2} R_\epsilon^{-\tau} + R_\epsilon^{1-\tau} \delta_2 + \delta_2 \right] |\xi - 2i|^{-\tau} \|U\| \leq C \delta_2 \|U\|_{\Omega_3}, \text{ for } R_\epsilon = O(\epsilon^{-2}).$$

□

Lemma 2.59. *Let $\mathcal{U}[M] = \frac{1}{\epsilon^2} g_1(\xi) \int_{-\infty}^{\xi} M(t) g_2(t) dt$, then $\|\mathcal{U}[M]\|_{\Omega_3} \leq C \frac{\epsilon^2}{|\nu|^2} \|U\|_{\Omega_3}$.*

Proof. The lemma follows from Lemma 2.50 and the proof of the above lemma. □

Proof. of Theorem 2.39: From (2.48), Lemmas 2.47-2.48, we obtain

$$(2.64) \quad \|U\|_{\Omega_3}^- \leq (\|\mathcal{U}[\bar{M}]\|_{\Omega_3} + \|\mathcal{U}[M]\|_{\Omega_3}) \leq C(\delta_2 + \frac{\epsilon^2}{|\nu|^2}) \|U\|_{\Omega_3}^-;$$

where C is some constant independent of ϵ , ν and δ_2 . From (2.53), when $\|F\|_0$ are small enough, $C(\delta_2 + \frac{\epsilon^2}{|\nu|^2}) < 1$ in (2.64). This implies $U(\xi) \equiv 0$ on Ω_3 and hence everywhere by analytic continuation. Hence $F(\xi) = F_+(\xi) = \hat{F}(\xi)$ and $F(\xi)$ is analytic in the upper half plane. Thus for $\xi \in \{Im \xi < 0\} \cap \mathcal{R}^-$, $I_2(\xi) = I_1(\xi)$, and equation (2.34) reduces to (2.27) in that region. □

2.3. Formulation of the half problem. If $F \in \mathbf{A}_0^-$ and satisfies symmetry condition $Im F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$, then Schwartz reflection principle applies and

$$(2.65) \quad F(\xi) = [F(-\xi^*)]^*, \text{ for } \xi \in \mathcal{R};$$

defines F in \mathcal{R}^+ ; consequently $F \in \mathbf{A}_0$ with $\|F\|_0 = \|F\|_0^-$. The reflection principle also implies

$$(2.66) \quad F'(\xi) = -[F'(-\xi^*)]^*, \text{ for } \xi \in \mathcal{R}.$$

For $F \in \mathbf{A}^-$, if we relax the symmetry condition $Im F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$, then it is still possible to define F and its derivative in \mathcal{R}^+ , based on F in \mathcal{R}^- using (2.65)-(2.66). However, this F in \mathcal{R}^+ is not the analytic continuation of F in \mathcal{R}^- since violation of symmetry condition implies that extension of F in \mathcal{R}^+ is discontinuous across $\{Re \xi = 0\} \cup \mathcal{R}$. Nonetheless, this still allows us to define analytic functions F_- in Ω_0 through (2.18), and F_+ in Ω_1 through (2.30), each of which have vanishing imaginary parts on the $Im \xi$ axis segment that are part of their domains of analyticity. Thus, I_2 is still defined as in (2.33) as an analytic function everywhere in \mathcal{R} . Also, the norms of these functions F_- , F_+ , and I_2 in their respective domains are completely controlled by $\|F\|_0^-$.

Half Problem: *Find function $F \in \mathbf{A}_{0,\delta}^-$ that are analytic in \mathcal{R}^- and continuous in its closure, and satisfies equation (2.34) in \mathcal{R}^- .*

Remark 2.60. If $F \in \mathbf{A}_\delta^-$ is a solution of Half Problem, and F satisfies

$$(2.67) \quad Im F = 0, \text{ on } \{Re \xi = 0\} \cap \mathcal{R};$$

then F is a solution to **Problem 2** and therefore the original Finger problem. Conversely, any solution F to **Problem 2** (and therefore the original Finger problem) is also a solution to the half problem.

3. SOLUTION TO THE HALF PROBLEM IN \mathcal{R}^-

In this section, by changes of variables, we first analyze equation (2.34) and identify possible singularity at $\xi = 0$ which corresponds to the finger tip. We then formulate an integral equation that is equivalent to (2.34). Near the singular point $\xi = 0$, we seek particular solution satisfying (3.33) and derive the differential and integral equations that govern the particular solution. By constructing a normal approximation sequence, we obtain existence of solution to the half problem.

3.1. Analysis of Equation (2.34). Let $F = \epsilon^2 W$, then (2.34) becomes

$$(3.1) \quad \epsilon^2 W'(\xi) = \frac{1}{2} \left\{ -[(W + I_2)^2 ((F^-)' + \bar{H}) + 2G_1(F^-)] + ((F^-)' + \bar{H}) \sqrt{(W + I_2)^2 - 4} \right\}$$

Let

$$(3.2) \quad Q(\xi) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(H - \bar{H})(t) dt}{H^{1/2}(t) \bar{H}^{1/2}(t)(t - \xi)} \text{ for } \text{Im } \xi < 0,$$

then

$$(3.3) \quad \begin{aligned} \tilde{I}(F) &= I_2(G) - Q(\xi) \\ &= -\frac{1}{2\pi} \int_{r_2}^{\infty} \frac{(F'_+ - F'_-) dt}{G_5(F_+, F_-)(t - \xi)} \\ &\quad - \frac{1}{2\pi} \int_{r_2}^{\infty} \frac{(H - \bar{H})}{\bar{H}^{1/2} \bar{H}^{1/2}} \frac{(-\epsilon^2 F'_+ F'_- - F'_+ \bar{H} - F'_- H) dt}{G_5(F_+, F_-)[H^{1/2} \bar{H}^{1/2} + G_5(F_+, F_-)](t - \xi)} \end{aligned}$$

$$(3.4) \quad G_5(F_+, F_-) = (F'_+ + H)^{1/2} (F'_- + \bar{H})^{1/2}.$$

Now let

$$(3.5) \quad V = W + Q$$

then (3.1) becomes

$$(3.6) \quad \epsilon^2 V' = \frac{-[V^2 \bar{H} + 2(H - \bar{H})] + \bar{H} V \sqrt{V^2 - 4}}{2} + \epsilon^2 Q' + G_6(V, F_-, \tilde{I}) + G_7(V, \tilde{I})$$

where

$$(3.7) \quad G_6(V, F_-, \tilde{I}) = F'_- \left[\frac{-(V + \tilde{I})^2 + (V + \tilde{I}) \sqrt{(V + \tilde{I})^2 - 4} + 2}{2} \right];$$

$$(3.8) \quad G_7(V, \tilde{I}) = -\frac{[2V\tilde{I} + (\tilde{I})^2] \bar{H}}{2} + \frac{1}{2} \bar{H} \frac{[2V\tilde{I} + (\tilde{I})^2][(V^2 - 4) + (V + \tilde{I})^2]}{(V + \tilde{I}) \sqrt{(V + \tilde{I})^2 - 4} + V \sqrt{V^2 - 4}}.$$

Let

$$(3.9) \quad V_0 = -\frac{2\gamma}{\sqrt{\xi^2 + \gamma^2}},$$

then

$$(3.10) \quad \sqrt{V_0^2 - 4} = -\frac{2i\xi}{\sqrt{\xi^2 + \gamma^2}},$$

and

$$(3.11) \quad -[V_0^2 \bar{H} + 2(H - \bar{H})] + \bar{H} V_0 \sqrt{V_0^2 - 4} = 0.$$

Let $p = V - V_0$ then p satisfies

$$(3.12) \quad \epsilon^2 p' + \tilde{Q}p = \epsilon^2(Q' - V_0') + G_6(p, \tilde{I}) + G_7(p, \tilde{I}) + G_8(p)$$

where

$$(3.13) \quad \tilde{Q}(\xi) = \frac{2H^{3/2}\bar{H}^{1/2}}{H + \bar{H}} = i \frac{(\xi + i\gamma)^{3/2}(\xi - i\gamma)^{1/2}}{\xi(\xi^2 + 1)},$$

$$(3.14) \quad G_8(p) = \frac{\frac{1}{2}\bar{H}p^2[(2V_0 + p)^2 + (2V_0^2 - 4)]}{(p + V_0)\sqrt{V_0^2 - 4 + (2V_0p + p^2)} + V_0\sqrt{V_0^2 - 4}} - \frac{\bar{H}p^2}{2} \\ - \frac{p^2(V_0^2 - 2)(2V_0 + p)[p^2 + 2V_0p + 2V_0^2 - 4]}{\left((p + V_0)\sqrt{V_0^2 - 4 + (2V_0p + p^2)} + V_0\sqrt{V_0^2 - 4}\right)^2 \sqrt{V_0^2 - 4}}.$$

where $G_6(p, \tilde{I})$ and $G_7(p, \tilde{I})$ are given by (3.7) and (3.8) with V being replaced by $p + V_0$.

Remark 3.1. It is to be noted that \tilde{Q} is singular at $\xi = 0$ which corresponds to the finger tip.

Let

$$(3.15) \quad N(p) = \epsilon^2(Q' - V_0') + G_6(p) + G_7(p) + G_8(p).$$

Lemma 3.2. F is a solution to the half problem in \mathcal{R}^- if and only if $p = \frac{1}{\epsilon^2}F + Q - V_0$ is a solution to (3.12) in \mathcal{R}^- .

Lemma 3.3. $p \in \mathbf{A}_0$ satisfies (3.12) if and only if p satisfies the following integral solution

$$(3.16) \quad p(\xi) = \mathcal{U}(N)[\xi] \equiv \frac{1}{\epsilon^2}g_1(\xi) \int_{-\infty}^{\xi} g_2(t)N(p)(t)dt.$$

Proof. Since p satisfies (3.12), then

$$(3.17) \quad p(\xi) = Cg_1(\xi) + \frac{1}{\epsilon^2}g_1(\xi) \int_{-\infty}^{\xi} g_2(t)N(p)(t)dt$$

for some constant C . Since $p(-\infty) = 0$, $g_1(-\infty) \neq 0$, Lemma 2.19 implies that $C = 0$. \square

In this section, we choose $\tau \geq 6/7$, $R_\epsilon = O(\epsilon^{-2}) > R$.

Let $\mathcal{R}_1 = \mathcal{R}^- \cap \{\xi : |\xi| \leq R_\epsilon\}$.

Lemma 3.4. Assume that $p \in \mathbf{A}_{0,\bar{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.18) \quad |G_k(p)[\xi]| \leq C\epsilon^2(1 + \|p\|_0) (1 + |\xi - 2i|^{-\tau}) |\xi - 2i|^{-1-\tau}.$$

where $k = 6, 7$ and C is independent of ϵ .

Proof. The lemma follows from (3.7),(3.8). \square

Lemma 3.5. Assume that $f, g \in \mathbf{A}_{0,\bar{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.19) \quad |G_k(f) - G_k(g)[\xi]| \leq C|\xi - 2i|^{-1-\tau}(\epsilon^2 + \bar{\delta}\epsilon^2|\xi - 2i|^{-\tau})\|f - g\|$$

where $k = 6, 7$ and C is independent of ϵ .

Proof. The lemma follows from (3.7),(3.8). \square

Lemma 3.6. Assume that $p \in \mathbf{A}_{0,\tilde{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.20) \quad |G_8(p)[\xi]| \leq C|\xi - 2i|^{-1-2\tau}\|p\|^2$$

where C is independent of ϵ .

Proof. The lemma follows from (3.14) □

Lemma 3.7. Assume that $f, g \in \mathbf{A}_{0,\tilde{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.21) \quad |G_8(f) - G_8(g)[\xi]| \leq C\tilde{\delta}|\xi - 2i|^{-1-2\tau}\|f - g\|$$

where C is independent of ϵ .

Proof. The lemma follows from (3.14) □

Lemma 3.8. Assume that $f \in \mathbf{A}_{0,\tilde{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.22) \quad |\mathcal{U}(f)[\xi]| \leq C(R_\epsilon^{-1+\tau} + \frac{\tilde{\delta}^2}{R_\epsilon^\tau \epsilon^2})|\xi - 2i|^{-\tau},$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.53-2.54, Lemma 3.3 and Lemma 3.5. □

Lemma 3.9. Assume that $f, g \in \mathbf{A}_{0,\tilde{\delta}}^-$. Then for $\xi \in \mathcal{R}_1$

$$(3.23) \quad |\mathcal{U}(f)[\xi] - \mathcal{U}(g)[\xi]| \leq C(\frac{1}{R_\epsilon^{1-\tau}} + \frac{\tilde{\delta}}{R_\epsilon^\tau \epsilon^2})\|f - g\|_0|\xi - 2i|^{-\tau},$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.53-2.54, Lemma 3.4 and Lemma 3.6. □

Let $\mathcal{R}_2 = \mathcal{R}^- \cap \{\xi : R \leq |\xi| \leq R_\epsilon\}$ and $R_\epsilon^l = r_l^- \cap \{\xi : |\xi| = R_\epsilon\}$. From Lemma 2.10 and 2.11, $\operatorname{Re} P(\xi)$ attain minimum at R_ϵ^l .

Define operator \mathcal{U}_2 as

$$(3.24) \quad \mathcal{U}_2[p] = \frac{1}{\epsilon^2} g_1(\xi) \int_{R_\epsilon^l}^\xi g_2(t) N(p)(t) dt$$

Lemma 3.10. If $f \in \mathbf{A}_0^-$, then $|N(f)(t)| \leq C\epsilon^2 (|t|^{-2} + |t|^{-(1+\tau)}\|f\|_0) + C\|f\|_0^2 |t|^{-1-2\tau}$, for $t \in \mathcal{R}_2$

Proof. The lemma follows from (3.7), (3.14) and (3.15). □

Lemma 3.11. If $f, g \in \mathbf{A}_0^-$, then for $t \in \mathcal{R}_2$

$$(3.25) \quad |N(f)(t) - N(g)(t)| \leq C\epsilon^2 |t|^{-1-\tau}\|f - g\|_0 + (\|g\| + \|f\|_0) |t|^{-1-2\tau}\|f - g\|_0.$$

Proof. The lemma follows from (3.7), (3.8), (3.14) and (3.15). □

Lemma 3.12. Assume that $f \in \mathbf{A}_{0,\tilde{\delta}}^-$. Then for $\xi \in \mathcal{R}_2$

$$(3.26) \quad |\xi - 2i|^\tau |\mathcal{U}_2(f)[\xi]| \leq C\epsilon^2 (R_\epsilon^\tau + R_\epsilon^1 \|f\|_0) + C\|f\|_0^2 R_\epsilon^{1-\tau},$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.55, Lemma 3.10 and Lemma 3.11. □

Lemma 3.13. Assume that $f, g \in \mathbf{A}_{0,\delta}^-$. Then for $\xi \in \mathcal{R}_2$

(3.27)

$$|\xi - 2i|^\tau |\mathcal{U}_2(f)[\xi] - \mathcal{U}_2(g)[\xi]| \leq C\epsilon^2 R_\epsilon^{(1-\tau)} \|f - g\|_0 + C(\|f\|_0 + \|g\|_0) R_\epsilon^{1-\tau} \|f - g\|_0,$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.55 and Lemma 3.10 and Lemma 3.11. \square

Let $\mathcal{R}_3 = \{|\xi| \leq R\} \cap \mathcal{R}^-$ and $R^l = r_l^- \cap \{\xi : |\xi| = R\}$.

Define operator \mathcal{U}_3 as

$$(3.28) \quad \mathcal{U}_3[f] = \frac{1}{\epsilon^2} g_1(\xi) \int_{R^l}^\xi g_2(t) N(f)(t) dt$$

Lemma 3.14. If $f \in \mathbf{A}_0^-$, then $|N(f)(t)| \leq C\epsilon^2 + C\|f\|_0^2$, for $t \in \mathcal{R}_3$

Proof. The lemma follows from (3.7), (3.14) and (3.15). \square

Lemma 3.15. If $f, g \in \mathbf{A}_0^-$ then for $t \in \mathcal{R}_3$

$$(3.29) \quad |N(f)(t) - N(g)(t)| \leq C\epsilon^2 \|f - g\|_0 + (\|g\|_0 + \|f\|_0) \|f - g\|_0.$$

Proof. The lemma follows from (3.7), (3.14) and (3.15). \square

Lemma 3.16. Assume that $f \in \mathbf{A}_{0,\delta}^-$. Then for $\xi \in \mathcal{R}_3$

$$(3.30) \quad |\mathcal{U}_3(f)[\xi]| \leq C\epsilon^2 + C\|f\|_0^2,$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.56 and Lemma 3.3 and Lemma 3.5. \square

Lemma 3.17. Assume that $f, g \in \mathbf{A}_{0,\delta}^-$. Then for $\xi \in \mathcal{R}_3$

$$(3.31) \quad |\mathcal{U}_3(f)[\xi] - \mathcal{U}_3(g)[\xi]| \leq C(\epsilon^2 + C(\|f\|_0 + \|g\|_0)) \|f - g\|_0.$$

where C is independent of ϵ .

Proof. The lemma follows from Lemma 2.19 and Lemma 3.4 and Lemma 3.6. \square

3.2. Analysis of (3.12) in a neighborhood of the origin. Let \mathcal{T} be the region bounded by $r_{u,3}$, negative imaginary axis, line segment $\{\xi : \xi = -\nu_1 + se^{-\pi i/6}, 0 \leq s \leq 2\sqrt{3}/3\nu_1\}$ and line segment $\{\xi : \xi = -\nu_1 + se^{\pi i/6}, 0 \leq s \leq 2\sqrt{3}/3\nu_1\}$ where ν_1 is as in Definition 1.1. \mathcal{T} is a neighborhood of $\xi = 0$ in \mathcal{R}^- .

In \mathcal{T} , we have from (2.32)

$$(3.32) \quad \tilde{Q}(\xi) = -\frac{\gamma^2}{\xi} + Q_1(\xi).$$

where $Q_1(\xi)$ is analytic at $\xi = 0$. We seek solution of the form

$$(3.33) \quad p = (\beta + q(\xi))\xi$$

where β is a nonzero constant that will be given later and $q(\xi)$ satisfies

$$(3.34) \quad |q(\xi)| = O(\sqrt{|\xi|}).$$

For $\xi \in \mathcal{T}$, using $F = \epsilon^2(p + V_0 - Q)$ we can write

$$(3.35) \quad F'_-(\xi) = F_{0,-}(\xi) + \epsilon^2 p'_-(\xi)$$

where

$$(3.36) \quad p'_-(\xi) = -\frac{1}{2\pi i} \int_{r_0^-} \frac{\bar{p}'(t) - \bar{p}'(0)}{t - \xi} dt - \frac{1}{2\pi i} \int_{r_0^-} \frac{[\bar{p}'(t)]^* - [\bar{p}'(0)]^*}{t^* + \xi} dt$$

$$(3.37) \quad F_{0,-}(\xi) = -\frac{\epsilon^2}{2\pi i} \int_{r_0} \frac{\bar{V}'_0(t) - \bar{Q}'(t)}{t - \xi} dt.$$

$$(3.38) \quad F'_+(\xi) = F_{0,+}(\xi) + \epsilon^2 p'_+(\xi)$$

where

$$(3.39) \quad p'_+(\xi) = \frac{1}{2\pi i} \int_{r_1^-} \frac{p'(t) - p'(0)}{t - \xi} dt + \frac{1}{2\pi i} \int_{r_1^-} \frac{[p'(t)]^* - [p'(0)]^*}{t^* + \xi} dt$$

$$(3.40) \quad F_{0,+}(\xi) = \frac{\epsilon^2}{2\pi i} \int_{r_1} \frac{V'_0(t) - Q'(t)}{t - \xi} dt.$$

Let

$$(3.41) \quad G_{5,0}(\xi) = (F'_{+,0} + H)^{1/2} (F'_{-,0} + \bar{H})^{1/2}.$$

$$(3.42) \quad G_{5,1}(p'_-, p'_+) = G_5(F'_-, F'_+) - G_{5,0}(\xi).$$

Let

$$(3.43) \quad \begin{aligned} \tilde{G}(F'_-, F'_+) &= \frac{(F'_+ - F'_-)}{G_5(F_+, F_-)} \\ &\quad - \frac{(H - \bar{H})}{H^{1/2} \bar{H}^{1/2}} \frac{(-\epsilon^2 F'_+ F'_- - F'_+ \bar{H} - F'_- H)}{G_5(F_+, F_-) [H^{1/2} \bar{H}^{1/2} + G_5(F_+, F_-)]} \end{aligned}$$

and

$$(3.44) \quad \tilde{G}_1(p'_-, p'_+) = \tilde{G}(F'_-, F'_+) - \tilde{G}(F'_{-,0}, F'_{+,0})$$

$$(3.45)$$

$$\tilde{I}(F) = -\frac{1}{2\pi} \int_{r_2} \frac{(\tilde{G}_1(p'_-, p'_+)[t] - \tilde{G}_1(p'_-, p'_+)[0])}{(t - \xi)} dt - \frac{1}{2\pi} \int_{r_2} \frac{\tilde{G}(F'_{-,0}, F'_{+,0})}{(t - \xi)} dt$$

Let

$$(3.46) \quad \tilde{I}_1(p)(\xi) = \tilde{I}(F)(\xi) - \tilde{I}(F)(0).$$

Plugging (3.33) into (3.7), (3.8) and (3.14), we have for $\xi \in B_\nu \cap \mathcal{R}^-$,

$$(3.47) \quad G_6(q, \xi) = \beta_{6,0}(p) + [F'_-(\xi) - F'_-(0)][G_{6,0}(\xi) + G_{6,1}(\xi, \tilde{I}_1, q)];$$

where

$$(3.48) \quad \beta_{6,0}(p) = [F'_{-,0}(0) + \epsilon^2 p'_-(0)] \left\{ 1 + \frac{1}{2} [-(-2 + \tilde{I}(0))^2 + (-2 + \tilde{I}(0)) \sqrt{-4\tilde{I}(0) + (\tilde{I}(0))^2}] \right\}.$$

$$(3.49) \quad G_7(q, \xi) = \beta_{7,0}(p) + \tilde{I}_1 \{ G_{7,0}(\xi) + G_{7,1}(\xi, \tilde{I}_1, q) \};$$

where

$$(3.50) \quad \beta_{7,0}(p) = -\frac{i\gamma}{2} [-4\tilde{I}(0) + (\tilde{I}(0))^2] - \frac{i\gamma}{2} (-2 + \tilde{I}(0)) \sqrt{-4\tilde{I}(0) + (\tilde{I}(0))^2}.$$

$$(3.51) \quad G_8(q) = -i\gamma\beta - i\gamma q + \sqrt{\xi}G_{8,1}(\sqrt{\xi}, q) + q^2G_{8,2}(\sqrt{\xi}, q);$$

where $G_{6,1}(\xi, \tilde{I}_1, q), G_{7,1}(\xi, \tilde{I}_1, q)$ are analytic in ξ, \tilde{I}_1 and q . $G_{8,1}(\xi, q), G_{8,2}(\xi, q)$ are analytic at $\xi = 0, q = 0$.

Let

$$(3.52) \quad \beta(p) = \frac{\epsilon^2(Q'(0) - V'_0(0)) + \beta_{6,0}(p) + \beta_{7,0}(p)}{\epsilon^2 - \gamma^2 + i\gamma}$$

Equation (3.12) becomes

$$(3.53) \quad \epsilon^2 q' - \frac{\gamma^2 - \epsilon^2 - i\gamma}{\xi} q = N_1(\xi, q);$$

where

$$(3.54) \quad \begin{aligned} N_1(\xi, p) = & \epsilon^2 Q_2(\xi) - Q_1(\xi)\beta - Q_1(\xi)q + \frac{F'_-(\xi) - F'_-(0)}{\xi} \left\{ G_{6,0}(\xi) + G_{6,1}(\xi, \tilde{I}_1, q) \right\} \\ & + \frac{\tilde{I}_1}{\xi} \left\{ G_{7,0}(\xi) + G_{7,1}(\xi, \tilde{I}_1, q) \right\} + \frac{1}{\sqrt{\xi}} G_{8,1}(\sqrt{\xi}, q) + \frac{q^2}{\xi} G_{8,2}(\sqrt{\xi}, q) \end{aligned}$$

where $Q_2(\xi)$ is given by

$$(3.55) \quad Q_2(\xi) = \frac{(Q'(\xi) - V'_0(\xi)) - (Q'(0) - V'_0(0))}{\xi}.$$

Let

$$(3.56) \quad h_1(\xi) = e^{-\frac{P_1(\xi)}{\epsilon^2}}, \quad h_2(\xi) = e^{\frac{P_1(\xi)}{\epsilon^2}}.$$

where $P_1(\xi)$ is

$$(3.57) \quad P_1(\xi) = (\gamma^2 - \epsilon^2 - i\gamma) \log \xi$$

and

$$\log \xi = \ln |\xi| + \arg \xi, \pi \leq \arg \xi \leq \frac{3\pi}{2}.$$

Remark 3.18. $\operatorname{Re} P_1(\xi)$ attains maximum in \mathcal{T} at $\xi = -i\frac{\sqrt{3}}{3}\nu_1$. For any $\xi \in \mathcal{T}$, there is a path $\mathcal{P}(\xi, -i\frac{\sqrt{3}}{3}\nu_1)$ from ξ to $-i\frac{\sqrt{3}}{3}\nu_1$ such that $\operatorname{Re} P_1(\xi)$ decreases from ξ to $-i\frac{\sqrt{3}}{3}\nu_1$ and $|\frac{d\operatorname{Re} P_1(\xi)}{d\xi}| \geq \frac{C}{|\xi|}$.

The integral form of (3.53) is

$$(3.58) \quad q = \tilde{q}(-i\frac{\sqrt{3}}{3}\nu_1)h_1(-i\frac{\sqrt{3}}{3}\nu_1)h_2(\xi) + h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^{\xi} h_1(t) \frac{N_1(t, q(t))}{\epsilon^2} dt.$$

where

$$(3.59) \quad \begin{aligned} \tilde{q}(-i\frac{\sqrt{3}}{3}\nu_1) &= \frac{p(-i\frac{\sqrt{3}}{3}\nu_1)}{-i\frac{\sqrt{3}}{3}\nu_1} - \beta \\ p(-i\frac{\sqrt{3}}{3}\nu_1) &= \mathcal{U}(N)[-i\frac{\sqrt{3}}{3}\nu_1] \equiv \frac{1}{\epsilon^2} g_1(-i\frac{\sqrt{3}}{3}\nu_1) \int_{-\infty}^{-i\frac{\sqrt{3}}{3}\nu_1} g_2(t) N(p)(t) dt. \end{aligned}$$

The space \mathbf{D} is defined as

$$(3.60) \quad \mathbf{D} = \{q(\xi) : q \text{ is analytic in } B_\nu \cap \mathcal{R}^- \text{ and bounded in its closure with } \sup_{\xi \in \overline{B_\nu \cap \mathcal{R}^-}} |\xi^{-1/2} q(\xi)| < \infty \}.$$

with norm

$$\|q\|_{\mathbf{D}} = \sup_{\xi \in \overline{B_\nu \cap \mathcal{R}^-}} |\xi^{-1/2} q(\xi)|$$

Lemma 3.19. *If $\xi^{1/2} n(\xi) \in C(\overline{B_\nu \cap \mathcal{R}^-})$, let $q_1(\xi) = \frac{1}{\epsilon^2} h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^\xi n(t) g_1(t) dt$, then $q_1 \in \mathbf{D}$ and $\|q_1\| \leq K_1 \max |\xi^{1/2} n(\xi)|$ for constant K_1 independent of ϵ and ν .*

Proof. We choose path $\mathcal{P}(\xi, -i\nu)$ as in remark 3.2,

$$\begin{aligned} |q_1(\xi)| &= \left| \frac{1}{\epsilon^2} \int_{-i\nu_1}^\xi n(t) \exp\left\{-\frac{\gamma^2 - \epsilon^2 - i\gamma}{\epsilon^2} (\log t - \log \xi)\right\} dt \right| \\ &\leq C \max |t^{1/2} n(t)| \int_{-i\frac{\sqrt{3}}{3}\nu_1}^\xi \left| t^{-1/2} \exp\left\{-\frac{\gamma^2 - \epsilon^2 - i\gamma}{\epsilon^2} (\log t - \log \xi)\right\} \right| |dt| \\ &\leq K_1 |\xi|^{1/2} \max |t^{1/2} n(t)| \end{aligned}$$

□

Lemma 3.20. *Let $q \in \mathbf{D}$, then $\max |\xi|^{1/2} |N_1(\xi, q(\xi))| \leq C_1 + C_2(\delta_1 + |\nu|) \|q\|_{\mathbf{D}}$.*

Proof. The lemma follows from (3.47)-(3.51) and (3.54). □

Lemma 3.21. *Let $q_1, q_2 \in \mathbf{D}$, then $\max |\xi|^{1/2} |N_1(\xi, q_1(\xi)) - N_1(\xi, q_2(\xi))| \leq C(\delta_1 + |\nu|) \|q_1 - q_2\|_{\mathbf{D}}$.*

Proof. The lemma follows from (3.47)-(3.51) and (3.54). □

3.3. Existence of solution to (3.12). In this section, we are going to prove the existence of solution $p(\xi)$ to (3.12) which satisfies (3.33) in \mathcal{T} .

From Definitions (3.16) (3.24) and (3.58), we can write

$$(3.61) \quad \mathcal{U}[p](\xi) = g_1(\xi) [g_2(R_\epsilon^l) \mathcal{U}(R_\epsilon^l) + g_2(\xi) \mathcal{U}_2[p](\xi)], \text{ for } \xi \in \mathcal{R}_2,$$

$$(3.62) \quad \mathcal{U}[p](\xi) = g_1(\xi) [g_2(R_\epsilon^l) \mathcal{U}(R_\epsilon^l) + g_2(R^l) \mathcal{U}_2[p](R^l) + g_2(\xi) \mathcal{U}_3[p](\xi)], \text{ for } \xi \in \mathcal{R}_3/\mathcal{T},$$

and

$$(3.63) \quad \mathcal{U}[p](\xi) = \xi(\beta[p] + \mathcal{U}_4[q](\xi)) \text{ for } \xi \in \mathcal{T}.$$

where

$$(3.64) \quad \mathcal{U}_4[q](\xi) = \left(-\beta[p] + \frac{\mathcal{U}(-i\frac{\sqrt{3}}{3}\nu_1)}{-i\frac{\sqrt{3}}{3}\nu_1} \right) h_1(-i\frac{\sqrt{3}}{3}\nu_1) h_2(\xi) + h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^\xi h_1(t) \frac{N_1(t, q(t))}{\epsilon^2} dt,$$

and β is given by (3.52).

In the following, we choose $\nu = O(\epsilon) < \nu_1$.

Lemma 3.22. *Let $p \in \mathbf{A}_0^-$, then for $\xi \in B_\nu \cap \mathcal{R}^-$*

$$(3.65) \quad \frac{|\tilde{G}_1(p'_-, p'_+)[\xi] - \tilde{G}_1(p'_-, p'_+)[0]|}{\sqrt{|\xi|}} \leq \epsilon^2 \left(K \sup_{\xi \in B_\nu \cap \mathcal{R}^-} \frac{|p'(\xi) - p'(0)|}{\sqrt{|\xi|}} + C \right);$$

and

$$(3.66) \quad \sup_{\xi \in r_2 \cap B_\nu} |\tilde{G}_1(p'_-, p'_+)| \leq \epsilon^2 \left(\frac{\|p\|_0}{\nu} + C \right).$$

Proof. The lemma follows from (3.43)-(3.44). \square

Lemma 3.23. *Let $p \in \mathbf{A}_0^-$, then for $\xi \in B_\nu \cap \mathcal{R}^-$*

$$(3.67) \quad \frac{|\tilde{I}(p)[\xi] - \tilde{I}(p)[0]|}{\sqrt{|\xi|}} \leq \epsilon^2 C \left(\sup_{\xi \in B_\nu \cap \mathcal{R}^-} \frac{|p'(\xi) - p'(0)|}{\sqrt{|\xi|}} + \nu^{-1/2} + \nu^{-3/2} \sup_{\xi \in \mathcal{R}^- / B_\nu} |p| \right);$$

and

$$(3.68) \quad C\epsilon^2 \leq |\tilde{I}(p)[0]| \leq \epsilon^2 C \left(\sqrt{\nu} \sup_{\xi \in B_\nu \cap \mathcal{R}^-} \frac{|p'(\xi) - p'(0)|}{\sqrt{|\xi|}} + 1 + \nu^{-1/2} |\log \nu| \sup_{\xi \in \mathcal{R}^- / B_\nu} |p| \right);$$

Proof. The lemma follows from Lemma 2.15, Lemma 2.31 and (3.45). \square

Lemma 3.24. *Let $p \in \mathbf{A}_0^-$, then*

$$(3.69) \quad C_1 \epsilon \leq |\beta(p)| \leq C \epsilon \left(1 + \left(\sqrt{\nu} \sup_{\xi \in B_\nu \cap \mathcal{R}^-} \frac{|p'(\xi) - p'(0)|}{\sqrt{|\xi|}} + \frac{|\log \nu|}{\nu^{1/2}} \sup_{\xi \in \mathcal{R}^- / B_\nu} |p| \right)^{1/2} \right);$$

where $C_1, C > 0$ are constants independent of ϵ and ν .

Proof. The lemma follows from (3.52) and the above lemma. \square

Let

$$(3.70) \quad \beta_0 = \beta(0), \quad p_0(\xi) = \frac{\beta_0 \xi}{\xi^2 + 1}, \quad q_0(\xi) = \frac{\beta_0 \xi^2}{\xi^2 + 1};$$

we define sequences $\{\beta_n\}, \{q_n(\xi)\}$ and $\{p_n(\xi)\}$ as follows:

$$(3.71) \quad \beta_n = \beta(p_{n-1});$$

$$(3.72) \quad \begin{aligned} q_n(\xi) &= \mathcal{U}_4[q_{n-1}] \\ &= \left(-\beta_{n-1} + \frac{\mathcal{U}[p_{n-1}](-i\frac{\sqrt{3}}{3}\nu_1)}{-i\frac{\sqrt{3}}{3}\nu_1} \right) h_1(-i\frac{\sqrt{3}}{3}\nu_1) h_2(\xi) \\ &\quad + h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^{\xi} h_1(t) \frac{N_1(t, q_{n-1}(t))}{\epsilon^2} dt; \end{aligned}$$

$$(3.73) \quad \begin{aligned} p_n(\xi) &= \xi(\beta_n + q_n(\xi)) \text{ for } \xi \in \mathcal{T}, \\ p_n(\xi) &= \mathcal{U}[p_{n-1}](\xi) \text{ for } \xi \in \mathcal{R}^- / \mathcal{T}. \end{aligned}$$

Lemma 3.25. *Let $h_3(\xi) = \mathcal{U}_4[0]$, then $h_3(\xi) \in \mathbf{D}$.*

Proof. The lemma follows from (3.64), Lemma 3.16, Lemma 3.19 and Lemma 3.33. \square

Let

$$(3.74) \quad \delta_2 = \|\mathcal{U}_4[0]\|, \delta_3 = \sup_{B_\nu \cap \mathcal{R}^-} |\xi^{1/2} N_1[0]|.$$

Lemma 3.26. *For sufficient small ϵ and ν , the following holds for all nonnegative integer n :*

$$(3.75) \quad |\beta_n| \leq C|\nu|^{1/4}$$

$$(3.76) \quad \|q_n\|_{\mathbf{D}} \leq 2\delta_2, \quad \sup_{B_\nu \cap \mathcal{R}^-} |\xi|^{1/2} |q'_n(\xi)| \leq 2\delta_3.$$

$$(3.77) \quad \epsilon^2 \sup_{B_\nu \cap \mathcal{R}^-} \frac{|p'_n(\xi) - p'_n(0)|}{|\xi|^{1/2}} \leq 2(\delta_2 + \delta_3).$$

$$(3.78) \quad \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau p_n| \leq C|\nu|^{1/2};$$

where $C > 0$ is independent of ϵ and ν .

Proof. We use induction to prove the lemma. The lemma holds for $n = 0$ from (3.71), assume that the lemma holds for all $n \leq k$. Then from (3.71) and Lemma 3.23, we obtain

$$\begin{aligned} |\beta_{k+1}| &= |\beta(p_k)| \leq C\epsilon^2 + C|\tilde{I}[p_k](0)|^{1/2} \\ &\leq C\epsilon^2 + C \left(\epsilon^2 \nu^{-1/2} |\log \nu| \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau p_k| + \sqrt{\nu} \epsilon^2 \sup_{B_\nu \cap \mathcal{R}^-} \frac{|p'_k(\xi) - p'_k(0)|}{|\xi|^{1/2}} \right)^{1/2} \\ &\leq C\epsilon^2 + (C\epsilon^2 |\log \nu| + C\sqrt{\nu}(\delta_2 + \delta_3))^{1/2} \leq C|\nu|^{1/4}; \end{aligned}$$

From (3.63), (3.74), Lemma 3.19 and Lemma 3.21, we obtain

$$\begin{aligned} |\xi|^{-1/2} |q_{k+1}| &\leq \left| |\xi|^{-1/2} \left(-\beta_k + \frac{\mathcal{U}_3[p_k](-i\frac{\sqrt{3}}{3}\nu_1)}{-i\frac{\sqrt{3}}{3}\nu_1} \right) h_1(-i\frac{\sqrt{3}}{3}\nu_1) h_2(\xi) \right| \\ &\quad + \left| |\xi|^{-1/2} h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^\xi h_1(t) \frac{N_1(t, q_k(t)) - N_1(t, 0)}{\epsilon^2} dt \right| + \|\mathcal{U}_4[0]\| \\ &\leq C|\xi|^{\frac{\gamma^2}{\epsilon^2}} + \|\mathcal{U}_4[0]\| \leq 2\delta_2; \end{aligned}$$

$$\begin{aligned} |\xi|^{1/2} |q'_{k+1}| &\leq \left| \frac{|\xi|^{-1/2}}{\epsilon^2} \left(-\beta_k + \frac{\mathcal{U}_3[p_k](-i\frac{\sqrt{3}}{3}\nu_1)}{-i\frac{\sqrt{3}}{3}\nu_1} \right) h_1(-i\frac{\sqrt{3}}{3}\nu_1) h_2(\xi) \right| \\ &\quad + \left| \frac{|\xi|^{-1/2}}{\epsilon^2} h_2(\xi) \int_{-i\frac{\sqrt{3}}{3}\nu_1}^\xi h_1(t) \frac{N_1(t, q_k(t))}{\epsilon^2} dt \right| + \left| \frac{|\xi|^{1/2} N_1(t, q_k(t))}{\epsilon^2} \right| \\ &\leq \frac{C}{\epsilon^2} |\xi|^{\frac{\gamma^2}{\epsilon^2}} + \frac{\sup_{B_\nu \cap \mathcal{R}^-} |\xi^{1/2} N_1[0]| + \|\mathcal{U}_4[0]\|}{\epsilon^2} \leq \frac{2\delta_2 + 2\delta_3}{\epsilon^2}; \end{aligned}$$

From (3.73) we have

$$\epsilon^2 \sup_{B_\nu \cap \mathcal{R}^-} \frac{|p'_{k+1}(\xi) - p'_{k+1}(0)|}{|\xi|^{1/2}} \leq \epsilon^2 |\xi|^{-1/2} |q_{k+1}| + \epsilon^2 |\xi|^{1/2} |q'_{k+1}| \leq 2(\delta_2 + \delta_3);$$

From (3.73) and Lemma 3.23 we have

$$\begin{aligned} \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau p_{k+1}| &\leq C\epsilon^2 \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau (Q' - V'_0)| + \\ &\quad C \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau (|G_6(p_k)| + |G_7(p_k)| + |G_8(p_k)|)| \\ &\leq C\epsilon^2 + C \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau (|F'_-(p_k)| + |\tilde{I}(p_k)| + |p_k^2|)| \\ &\leq C\epsilon^2 + C\sqrt{\nu}\epsilon^2 \sup_{B_\nu \cap \mathcal{R}^-} \frac{|p'_k(\xi) - p'_k(0)|}{|\xi|^{1/2}} \\ &\quad + C\epsilon^2 |\log |\nu|| \sup_{\mathcal{R}^-/B_\nu} |(\xi + 2i)^\tau p_k| + C\nu \leq C|\nu|^{1/2}; \end{aligned}$$

□

Theorem 3.27. *For sufficient small ϵ and $\nu = O(\epsilon)$, there exist subsequences $\{\beta_{n_k}\}, \{p_{n_k}(\xi)\}$ such that $\lim \beta_{n_k} = \beta, \lim p_{n_k}(\xi) = p(\xi)$ in \mathbf{A}_0^- , and p is a solution of (3.12). Hence $F = \epsilon^2(p - Q + V_0)$ is a solution of the half problem .*

Proof. From above lemma, β_n is a bounded sequence and $p_n(\xi)$ is uniformly bounded and $\{p_n(\xi)\}$ is a normal family. From Montel's Theorem, there exist subsequences $\{\beta_{n_k}\}, \{p_{n_k}(\xi)\}$ such that $\lim \beta_{n_k} = \beta, \lim p_{n_k}(\xi) = p(\xi)$ pointwisely. Since \mathbf{A}_0^- is Banach space, $p \in \mathbf{A}_0^-$. The theorem then follows from Lemma 3.2 and Lemma 3.3. □

4. SELECTION OF FINGER WIDTH: ANALYSIS NEAR $\xi = -i$

4.1. Derivation of Equation Near $\xi = -i$. In order to investigate whether or not the symmetry condition $\text{Im } F = 0$ on $\{Re \xi = 0\} \cap \mathcal{R}$ is satisfied, it is necessary to investigate a neighborhood of a turning point ($\xi = -i\gamma$ in our formulation), as first suggested from formal calculations of Combescot *et al* (1986). To that effect, we rewrite

$$(4.1) \quad F(\xi) = \epsilon^2 I_2(\xi) + \frac{i\epsilon^2 [(F'(\xi) + H) - (F'_-(\xi) + \bar{H})]}{(F'(\xi) + H)^{1/2} (F'_-(\xi) + \bar{H})^{1/2}};$$

We introduce

$$(4.2) \quad a = \frac{\gamma - 1}{\epsilon^{4/3}};$$

$$(4.3) \quad \xi = -i + i2^{1/3}\epsilon^{4/3}y^2, \quad G(y) = -i2^{-1/3} \left(i2^{1/3}y^2 F'(\xi) - \frac{1}{2}(2^{1/3}y^2 + a) \right)^{-1/2};$$

then (4.1) becomes:

$$(4.4) \quad \frac{dG}{dy} + \frac{1}{yG^2} = -y - \frac{\bar{\delta}_1}{y} + \epsilon^{2/3} E_2(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1})$$

where

$$(4.5) \quad \bar{\delta}_1 = \frac{a}{2^{1/3}};$$

where $E_2(\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1})$ is analytic function of $\epsilon^{2/3}, \epsilon^{2/3}y, G, G', y^{-1}$.

Note that the leading order equation obtained from dropping ϵ terms in (4.4) is similar to equation (133) in Chapman and King (2003). In order to get the equation close to the normal form discussed of Costin (1998), it is convenient to introduce additional change in variables:

$$(4.6) \quad \eta = \frac{2}{3}y^3, \quad \psi(\eta) = 1 - yG(y);$$

then (4.4) becomes:

$$(4.7) \quad \frac{d\psi}{d\eta} + \psi = -\frac{1}{3\eta} - \frac{1}{3\eta}\psi + \frac{a}{6^{2/3}\eta^{2/3}} + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n + \epsilon^{2/3} E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \eta^{-2/3}),$$

where $E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \eta^{-2/3})$ is analytic in $\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \eta^{-2/3}$ with a series representation convergent for small values of each argument.

It is to be noted

$$E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \eta^{-2/3}) = \sum_m^{\infty} E_m(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \eta^{-2/3}) \psi^m$$

and since each of the arguments for E_m can be safely be assumed to be in a compact set, it follows that there exists numbers A, ρ_2 are each independent of any parameter so that

$$(4.8) \quad |E_m| < A\rho_2^m$$

Theorem 4.1. *Let $F(\xi)$ be the solution of the Weak problem as in Theorem ?? . After change of variables:*

$$(4.9) \quad \xi = -i + i\epsilon^{4/3} \frac{3^{2/3}}{2^{1/3}} \eta^{2/3};$$

$$(4.10) \quad \psi(\eta, \epsilon, a) = - \left(1 + \frac{2^{4/3}a}{3^{2/3}\eta^{2/3}} - 2iF'(\xi(\eta)) \right)^{-1/2} + 1;$$

$\psi(\eta, \epsilon, a)$ satisfies equation (4.7) for $k_0\epsilon^{-2} \leq |\eta| \leq k_1\epsilon^{-2}, 0 \leq \arg \eta < \frac{3\pi}{4}$ (where k_0 and k_1 are some constants independent of ϵ) and the asymptotic condition

$$(4.11) \quad \psi(\eta, \epsilon, a) \rightarrow 0, 0 \leq \arg \eta < \frac{3\pi}{4}, \text{ as } \epsilon \rightarrow 0;$$

in that domain.

Proof. Since $\eta = O(\epsilon^{-2})$ in the given domain, using (4.9), we have $|\xi + i| = O(1)$ as $\epsilon \rightarrow 0$. Applying Theorem 3.27 and transformations (4.9) and (4.10), (2.35) implies (4.7). For $\xi \in \mathcal{R}^-, \frac{\pi}{2} < \arg(\xi + i) < \pi$, which on using transformation (4.9) implies $0 < \arg \eta < \frac{3\pi}{4}$. Continuity implies that (4.7) is satisfied for $\arg \eta = 0$ as well. Since $F'(\xi) \sim O(\epsilon)$, and $\eta^{-2/3} \sim O(\epsilon^{4/3})$, using (4.10), we obtain (4.11). \square

4.2. Leading Inner problem analysis. Setting $\epsilon = 0$ in equation (4.7), we get the leading order equation:

$$(4.12) \quad \frac{d\psi}{d\eta} + \psi = -\frac{1}{3\eta} - \frac{1}{3\eta}\psi + \frac{a}{6^{2/3}\eta^{2/3}} + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n$$

with far-field matching condition:

$$(4.13) \quad \psi(\eta, a) \rightarrow 0, \text{ as } |\eta| \rightarrow \infty, 0 \leq \arg \eta < \frac{3\pi}{4};$$

We shall prove the following theorem:

Theorem 4.2. *There exists large enough $\rho_0 > 0$ such that (4.12), (4.13) have a unique analytic solution $\psi_0(\eta, a)$ in the region $|\eta| \geq \rho_0, \arg \eta \in (-\frac{\pi}{8}, \frac{3\pi}{4})$.*

The proof of this Theorem will be given after some definitions and lemmas.

Definition 4.3. the region

$$\mathcal{R}_1 = \{\eta : |\eta| > \rho_0, \arg \eta \in (-\frac{\pi}{8}, \frac{3\pi}{4})\}$$

for some large ρ_0 independent of ϵ .

Definition 4.4.

$$(4.14) \quad \psi_1(\eta) = e^{-\eta}, \psi_2(\eta) = e^{\eta};$$

$\psi_1(\eta)$ satisfy the following equation exactly:

$$(4.15) \quad \mathcal{L}\psi \equiv \frac{d\psi}{d\eta} + \psi = 0;$$

Equation (4.12) can be rewritten as

$$(4.16) \quad \mathcal{L}\psi = \mathcal{N}_1(\eta, \psi) \equiv -\frac{1}{3\eta} - \frac{1}{3\eta}\psi + \frac{a}{6^{2/3}\eta^{2/3}} + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n.$$

We consider solution ψ of the following integral equation:

$$(4.17) \quad \psi = \mathcal{L}_1\psi \equiv \psi_1(\eta) \int_{\infty e^{3\pi i/4}}^{\eta} \psi_2(t) \mathcal{N}_1(t, \psi(t)) dt$$

Definition 4.5.

$$(4.18) \quad \mathbf{B}_1 = \left\{ \psi(\eta) : \psi(\eta) \text{ analytic in } \mathcal{R}_1 \right. \\ \left. \text{and continuous on } \overline{\mathcal{R}_1}, \sup_{\mathcal{R}_1} |\eta^{2/3} \psi(\eta)| < \infty \right\}$$

\mathbf{B}_1 is Banach space with norm

$$(4.19) \quad \|\psi\| = \sup_{\mathcal{R}_1} |\eta^{2/3} \psi(\eta)|;$$

Lemma 4.6. *Let $\mathcal{N} \in \mathbf{B}_1$, then*

$$\phi_1(\eta) := \psi_1(\eta) \int_{\infty e^{i3\pi/4}}^{\eta} \psi_2(t) \mathcal{N} dt \in \mathbf{B}_1,$$

and $\|\phi_1\| \leq K \|\mathcal{N}\|$, where K is independent of ρ_0 .

Proof. For $\eta \in \mathcal{R}_1$, we use straight lines in the t -plane to connect η to $\infty e^{i5\pi/8}$ so $\operatorname{Re} t$ is increasing monotonically from $\infty e^{i5\pi/8}$ to η and on that path, characterized by arc-length s , $\frac{d}{ds} \operatorname{Re} t(s) > C > 0$. Further, $C_1|\eta| \leq |t|$ for nonzero C_1 . Then,

$$\begin{aligned} |\phi_1(\eta)| &= \left| \int_{\infty e^{i3\pi/4}}^{\eta} e^{\eta-t} \mathcal{N} dt \right| \\ &\leq C \int_{\infty e^{i3\pi/4}}^{\eta} |t|^{-2/3} |e^{\eta-t}| |t^{2/3} \mathcal{N}| |dt| \\ &\leq \frac{C \|\mathcal{N}\|}{|\eta|^{2/3}} \int_{\infty e^{i3\pi/4}}^{\eta} |e^{\eta-t}| |dt| \leq \frac{C \|\mathcal{N}\|}{|\eta|^{2/3}} \end{aligned}$$

□

Definition 4.7. Define $T_1(\eta)$ so that $T_1(\eta) := \mathcal{L}_1 0$.

Remark 4.8. Since $|\eta^{2/3} \mathcal{N}_1(\eta, 0)|$ is bounded, Lemma 4.6 implies $T_1 \in \mathbf{B}_1$.

Definition 4.9. $\sigma_1 = \|T_1\|$; $\mathbf{B}_{\sigma_1} := \{\psi \in \mathbf{B}_1 : \|\psi\| \leq 2\sigma_1\}$

Lemma 4.10. If $\psi \in \mathbf{B}_{\sigma_1}$, $\phi \in \mathbf{B}_{\sigma_1}$, then $\mathcal{N}_1(\eta, \psi) \in \mathbf{B}_1$ and

$$\begin{aligned} \|\mathcal{N}_1(\eta, \psi)\| &\leq 2K_1\sigma_1 \left[\rho_0^{-2/3} \sigma_1 + \rho_0^{-1} \right] + \left[\frac{1}{3\rho_0} + \frac{2^{4/3}a}{3^{2/3}} \right] \\ \|\mathcal{N}_1(\eta, \psi) - \mathcal{N}_1(\eta, \phi)\| &\leq K_1 \left[\rho_0^{-2/3} \sigma_1 + \rho_0^{-1} \right] (\|\phi - \psi\|) \end{aligned}$$

for some numerical constant K_1 and for $8\sigma_1\rho_0^{-2/3} < 1$.

Proof. It is clear from (4.16) that

$$(4.20) \quad |\mathcal{N}_1(\eta, \psi) - \mathcal{N}_1(\eta, \phi)| \leq \frac{|\psi(\eta) - \phi(\eta)|}{4\|\eta\|^2} + \frac{1}{2} \sum_{k=2}^{\infty} (k+1) |\psi^k - \phi^k|$$

Noting, $|\psi| \leq 2\sigma_1|\eta|^{-2/3}$, $|\phi| \leq 2\sigma_1|\eta|^{-2/3}$ and from simple induction,

$$|\psi^k - \phi^k| \leq k(|\psi| + |\phi|)^{k-1} |\psi - \phi|, \text{ for } k \geq 1,$$

we obtain for $\sigma_1\rho_0^{-2/3} < \frac{1}{8}$,

$$(4.21) \quad \|\mathcal{N}_1(\eta, \psi) - \mathcal{N}_1(\eta, \phi)\| < \left[\frac{1}{3\rho_0} + \frac{K_2\sigma_1}{\rho_0^{2/3}} \right] \|\psi - \phi\|$$

for some numerical constant K_2 . On the other hand,

$$\|\mathcal{N}_1(\eta, 0)\| \leq \frac{1}{3\rho_0^{2/3}} + \frac{2^{4/3}a}{3^{2/3}}$$

So, it is clear from adding the two results above (with $\phi = 0$), it follows that that for $\psi \in \mathbf{B}_{\sigma_1}$,

$$\|\mathcal{N}_1(\eta, \psi)\| < \left[\frac{1}{3\rho_0^{2/3}} + \frac{2^{4/3}a}{3^{2/3}} \right] + \sigma_1 \left[\frac{1}{4\rho_0} + \frac{K_2\sigma_1}{\rho_0^{2/3}} \right]$$

□

Lemma 4.11. For sufficiently large ρ_0 , the operator \mathcal{L}_1 as defined in (4.17) is a contraction from \mathbf{B}_{σ_1} to \mathbf{B}_{σ_1} ; hence there is a unique solution ψ in this function space.

Proof. From Lemma 4.6 and Lemma 4.10:

$$\begin{aligned}\|\mathcal{L}_1\psi - T_1\| &\leq \|\mathcal{L}_1\psi - \mathcal{L}_1 0\| \leq 2K\|\mathcal{N}_1(\eta, \psi) - \mathcal{N}_1(\eta, 0)\| \\ &\leq 2KK_1 \left[\rho_0^{-2/3}\sigma_1 + \rho^{-1} \right] \|\psi\|\end{aligned}$$

$$\|\mathcal{L}_1\psi - \mathcal{L}_1\phi\| \leq 2K\|\mathcal{N}_1(\eta, \psi) - \mathcal{N}_1(\eta, \phi)\| \leq 2KK_1[\rho_0^{-2/3}\sigma_1 + \rho^{-1}]\|\psi - \phi\|;$$

So,

$$\|\mathcal{L}_1\psi\| \leq \|\mathcal{L}_1\psi - \mathcal{L}_1 0\| + \|T_1\| \leq 2\sigma_1$$

for sufficiently large ρ_0 . \square

Remark 4.12. It is easy to see that the previous lemma holds when we change the restriction on $\arg \eta$ in the definition of \mathcal{R}_1 to $(0, \frac{3\pi}{4})$. This comment is relevant to the following lemma.

Lemma 4.13. *Any solution ψ to (4.12) satisfying condition (4.13) in the domain \mathcal{R}_1 must be in \mathbf{B}_{σ_1} and satisfy integral equation: $\psi = \mathcal{L}_1\psi$ for sufficiently large ρ_0*

Proof. First, we note that if we use variation of parameter, the most general solution to (4.11) satisfies the integral equation

$$\psi = \mathcal{L}_1\psi + C_1\psi_1$$

Now, if we assume $\|\psi\|_\infty$ to be small, as implied by condition (4.13), when ρ_0 is chosen large, it follows from inspection of the the right hand side of (4.12) that $\|\mathcal{N}_1(\eta, \psi)\|_\infty$ is also small. Since Lemma 4.6 is easily seen to hold when the norm is replaced by $\|\cdot\|_\infty$, it follows that $\mathcal{L}_1\psi$ is also small. However, $C_1\psi_1(\eta)$ is unbounded in \mathcal{R}_1 unless $C_1 = 0$. Therefore, any solution to (4.12) satisfying condition (4.13) must satisfy integral equation $\psi = \mathcal{L}_1\psi$. If we were to use the norm $\|\cdot\|_\infty$ instead of the the weighted norm $\|\cdot\|$ in the definition of the Banach Space \mathbf{B}_1 , it is easily seen that each of Lemmas 4.6-4.11 would remain valid for small enough σ_1 , as appropriate when condition (4.13) holds and ρ_0 is large. Thus, it can be concluded that the solution to $\psi = \mathcal{L}_1\psi$ is unique in the bigger space of functions for which ψ satisfies (4.13) and ρ_0 is chosen large enough. However, from previous Lemma 4.11, it follows that this unique solution must be in the function space \mathbf{B}_1 and therefore satisfies $\psi = O(\eta^{-2/3})$ for large η . \square

Proof of Theorem 4.2 follows immediately from Lemmas 4.11 and 4.13

Theorem 4.14. *If $\psi_0(\eta, a)$ is the solution in Theorem 4.2, then*

Im $\psi_0(\eta, a) = S(a)e^{-\eta}(1 + o(1))$ on the real η axis and $\eta \rightarrow \infty$.

Proof. Plugging $\psi_0 = Re \psi_0 + iIm \psi_0$ in equation (4.12), then taking imaginary part, then $Im \psi_0(\eta)$ satisfies the following linear homogeneous equation on real positive η axis:

$$(4.22) \quad \frac{d Im \psi_0}{d\eta} + \left(1 + \frac{1}{3\eta} + E(\eta)\right) Im \psi_0 = 0;$$

where $E(\eta)$ is obtained from an homogeneous expression of $Re \psi_0$ and $Im \psi_0$. Since *a priori* both $Re \psi_0 \sim O(\eta^{-2/3})$ and $Im \psi_0 \sim O(\eta^{-2/3})$ as $\eta \rightarrow \infty$, we obtain

$$(4.23) \quad E(\eta) \sim O(\eta^{-2/3}), \text{ as } \eta \rightarrow \infty;$$

There is constant $S(a)$ so that $Im \psi_0(\eta) = S(a)e^{-(1+\frac{1}{3\eta}+E(\eta))\eta} = S(a)e^{-\eta}(1+o(\eta))$. \square

Remark 4.15. From Theorem 4.14, $Im \psi_0(\eta, a) = 0$ iff $S(a) = 0$. Previous numerical results and formal asymptotic results (Chapman and King 2003) suggest that $S(a) = 0$ if and only if a takes on a discrete set of values.

4.3. Full Inner Problem Analysis. Now we go back to the full inner equation (4.7). From Theorem 4.1, (4.7) with matching condition (4.11) has unique solution in the domain $k_1\epsilon^{-2} \geq |\eta| \geq k_0\epsilon^{-2}$, $\arg \eta \in (0, \frac{3\pi}{4})$. We shall first prove that this solution can be extended to the region:

$\mathcal{R}_2 = \{\eta : \rho_0 < Im \eta + Re \eta < \tilde{k}_0\epsilon^{-2}, \arg \eta \in [0, \frac{3\pi}{4}); -Im \eta + \rho_0 < Re \eta < Im \eta + \tilde{k}_0\epsilon^{-2}, \arg \eta \in (-\frac{\pi}{8}, 0]\}$, where $k_0 < \tilde{k}_0 < k_1$.

Definition 4.16. Let $\psi = \tilde{\psi}(\eta)$ be the unique analytic solution in Theorem 4.1 for $|\eta| \geq k_0\epsilon^{-2}$, $\arg \eta \in (0, \frac{3\pi}{4})$, restricted to the line segment $\{\eta : Im \eta + Re \eta = \tilde{k}_0\epsilon^{-2}, \arg \eta \in [0, \frac{3\pi}{4})\}$.

Definition 4.17.

$$(4.24) \quad \eta_0 = \tilde{k}_0\epsilon^{-2}, \quad \eta_1 = \tilde{k}_0\epsilon^{-2} \frac{1}{\sin \frac{\pi}{4} - 1} e^{\frac{3i\pi}{4}}, \quad \eta_2 = i\tilde{k}_0\epsilon^{-2};$$

Equation (4.7) can be rewritten as

$$(4.25) \quad \begin{aligned} \mathcal{L}\psi &= \mathcal{N}_2(\eta, \epsilon, \psi) \\ &\equiv -\frac{1}{3\eta} - \frac{1}{3\eta}\psi + \frac{a}{6^{2/3}\eta^{2/3}} \\ &\quad + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^n (n+1) \psi^n + \epsilon^{2/3} E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \eta^{-2/3}); \end{aligned}$$

We consider solution ψ of the following integral equation:

$$(4.26) \quad \psi = \mathcal{L}_2\psi \equiv \psi_1(\eta) \int_{\eta_1}^{\eta} \psi_2(t) \mathcal{N}_2(t, \psi) dt + \tilde{\psi}(\eta_1)$$

Definition 4.18.

$$(4.27) \quad \mathbf{B}_2 = \{\psi(\eta) : \psi(\eta) \text{ analytic in } \mathcal{R}_2, \text{ and continuous on } \overline{\mathcal{R}_2}\}$$

\mathbf{B}_2 is Banach space with norm

$$(4.28) \quad \|\psi\| = \sup_{\overline{\mathcal{R}_2}} \rho_1 |\psi(\eta)|, \text{ where } \rho_1 = \rho_0^{2/3};$$

Lemma 4.19. Let $\mathcal{N} \in \mathbf{B}_2$. Then

$$\phi_1(\eta) := \psi_1(\eta) \int_{\eta_1}^{\eta} \psi_2(t) \mathcal{N}(t) dt \in \mathbf{B}_2;$$

and $\|\phi_1\| \leq K \|\mathcal{N}\|$, where K is a numerical constant independent of any parameters.

Proof. For $\eta \in \mathcal{R}_2$, we use straight lines to connect η_1 so that $\operatorname{Re} t$ is increasing from η_1 to η and $C_1|\eta| \leq |t| \leq C_2|\eta|$.

$$\begin{aligned} |\phi_1(\eta)| &= \left| \int_{\eta_1}^{\eta} e^{-\eta+t} \mathcal{N} dt \right| \\ &\leq \frac{K}{\rho_0^{2/3}} \int_{\eta_1}^{\eta} |e^{-\eta+t}| |\rho_0^{2/3} \mathcal{N}| |dt| \\ &\leq \frac{K \|\mathcal{N}\|}{\rho_0^{2/3}} \int_{\eta_1}^{\eta} |e^{\eta-t}| |dt| \leq \frac{K \|\mathcal{N}\|}{\rho_0^{4/3}} \end{aligned}$$

□

Definition 4.20. $T_2(\eta) := \mathcal{L}_2 0$, $\sigma_2 \equiv \|T_2\|$.

Remark 4.21. Since $\mathcal{N}_2(\eta, \epsilon, 0, 0) = O(\eta^{-4/3}, \epsilon^{2/3})$ in \mathcal{R}_2 , it follows from Lemma 4.19 that $\sigma_2 = O(1)$, as $\epsilon \rightarrow 0^+$.

We define space

$$\mathbf{B}_{\sigma_2} = \{\psi \in \mathbf{B}_2 : \|\psi\| \leq 2\sigma_2\}$$

Lemma 4.22. *If $\psi \in \mathbf{B}_{\sigma_2}$, $\phi \in \mathbf{B}_{\sigma_2}$, then for $\rho_1 > \max\{4\sigma_2\rho_2, 8\sigma_2\}$, ρ_2 as defined in equation (4.8),*

$$\|\mathcal{N}_2(\eta, \epsilon, \psi) - \mathcal{N}_2(\eta, \epsilon, \phi)\| \leq \left[\frac{1}{3\rho_0} + \frac{K_2\sigma_1}{\rho_0^{2/3}} + 8A\sigma_2\epsilon^{2/3} \right] \|\phi - \psi\|,$$

for sufficiently small ϵ , where K_2 is a numerical constant.

Proof. Using (4.25), $|\psi| \leq 2\sigma_2\rho_1^{-1}$, $|\phi| \leq 2\sigma_2\rho_1^{-1}$ and inequality

$$|\psi^k - \phi^k| \leq k(|\psi| + |\phi|)^{k-1} |\psi - \phi|, \text{ for } k \geq 2$$

we have from (4.8)

$$\begin{aligned} &\epsilon^{2/3} |E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \psi, \eta^{-2/3}) - E_3(\epsilon^{2/3}, \epsilon^{2/3}\eta^{2/3}, \phi, \eta^{-2/3})| \\ &\leq \frac{8A\rho_2\epsilon^{2/3}}{\rho_1} \|\phi - \psi\| \end{aligned}$$

Combining with (4.21), the lemma follows. □

Theorem 4.23. *There exists a unique solution $\psi \in \mathbf{B}_{\sigma_2}$ of equation (4.2) for all sufficiently large ρ_0 and small ϵ .*

Proof. Using (4.26), Lemma 4.19 and Lemma 4.22, it is easily seen that

$$\begin{aligned} \|\mathcal{L}_2(u)\| &\leq \|\mathcal{L}_2(u) - \mathcal{L}_2(0)\| + \|T\|_2 \\ &\leq \sigma_2 + 4\sigma_2 K \left[\frac{1}{3\rho} + \frac{\sigma_1 K_2}{\rho_0^{2/3}} + 8A\sigma_2\epsilon^{2/3} \right] < 2\sigma_2 \end{aligned}$$

for sufficiently large ρ_0 and small ϵ .

On the other hand,

$$\|\mathcal{L}_2(u_1) - \mathcal{L}_2(u_2)\| \leq K \left[\frac{1}{3\rho_0} + \frac{\sigma_1 K_2}{\rho_0^{2/3}} + 8A\sigma_2\epsilon^{2/3} \right] \|(u_1 - u_2)\|.$$

Hence the proof follows from contraction mapping theorem on a Banach space. □

Lemma 4.24. *Let $\psi(\eta)$ be the solution of (4.26) as in Theorem 4.23 then $\psi(\eta)$ is a solution of (4.25) with $\psi(t) \equiv \tilde{\psi}(t)$ for $t \in \{\eta : \operatorname{Re} \eta + \operatorname{Im} \eta = \tilde{k}_0 \epsilon^{-2}, \arg \eta \in [0, \frac{3\pi}{4}]\}$.*

Proof. Since $\tilde{\psi}(t)$ is a solution of (4.25) for $t \in \{\eta : \operatorname{Re} \eta + \operatorname{Im} \eta = \tilde{k}_0 \epsilon^{-2}, \arg \eta \in [0, \frac{3\pi}{4}]\}$, by variation of parameters:

$$(4.29) \quad \tilde{\psi}(t) = \mathcal{L}_2 \tilde{\psi} + \tilde{\psi}(\eta_1)$$

By equation (4.26), we have

$$(4.30) \quad \psi(t) - \tilde{\psi}(t) = \psi_1(t) \int_{\eta_1}^t \psi_2(t) \left(\mathcal{N}_2(\psi) - \mathcal{N}_2(\tilde{\psi}) \right) dt$$

Using Lemma 4.19 and Lemma 4.22, we have

$$\|\psi - \tilde{\psi}\| \leq K \left[\frac{1}{3\rho^1} + \frac{\sigma_1 K_2}{\rho_0^{2/3}} + 8A\sigma_2 \epsilon^{2/3} \right] \|(\psi - \tilde{\psi})\|$$

for sufficiently large ρ_0 and small ϵ_0 . So $\psi(t) \equiv \tilde{\psi}(t)$. \square

Theorem 4.25. (1) *For large enough ρ_0 , there exists a unique solution $\psi(\eta, \epsilon, a)$ of (4.7), (4.11) in region $R_1 \geq |\eta| \geq \rho_0$ for $\arg \eta \in [0, 3\pi/4]$, where R_1 is some constant chosen to be independent of ϵ .*

(2) *The solution $\psi(\rho_0, \epsilon, a)$ of (1) is analytic in $\epsilon^{2/3}, a$, as $\epsilon \rightarrow 0$.*

(3) *Furthermore, $\lim_{\epsilon \rightarrow 0^+} \psi(\eta, \epsilon, a) = \psi_0(\eta, a)$ for $R_1 \geq |\eta| \geq \rho_0$.*

Proof. Part (1) follows from Theorem 4.1 and Theorem 4.23 and Theorem 4.24. Note that as $\epsilon \rightarrow 0$, $\epsilon^{2/3} E_3(\epsilon, \eta, \psi) \rightarrow 0$ uniformly in the region given. Part (2) follows from the theorem of dependence of solution on parameters (see, for instance, Theorem 3.8.5 in Hille 1976) \square

Lemma 4.26. *Let $F(\xi)$ be the solution of the half problem in Theorem 3.27, we define $q(\xi)$ so that $q(\xi) = \frac{F(\xi) - [F(-\xi^*)]^*}{2i}$ (Note this is the same as $\operatorname{Im} F$ on $\{\operatorname{Re} \xi = 0\} \cap \mathcal{R}$). Then q satisfies the following homogeneous equation on imaginary ξ axis: $\{\xi = is\}$:*

$$(4.31) \quad \epsilon^2 \frac{dq}{ds} + (2iH(is)Q(is) + \tilde{L}(s))q = 0;$$

where $\tilde{L}(s)$ is some real function and $\tilde{L}(s) \sim O(\epsilon^2)$ as $\epsilon \rightarrow 0$.

Proof. On imaginary axis $\{\xi = is\}$: Using (2.11), $L(is) = \frac{\sqrt{\gamma^2 - s^2}(s + \gamma)}{(1 - s^2)^2}$ is real. Using (2.58), $i(\tilde{H} - H)(is) = \frac{2\gamma}{(1 - s^2)}$ is real. By taking imaginary part in equation (2.42), we have the lemma. \square

Remark 4.27. F is analytic in $[-i + i\epsilon^{2/3}\rho_0, -ib)$.

Lemma 4.28. *If $q(is_1) = 0$ with $-1 + \epsilon^{4/3}\rho_0 \leq s_1 < -b$, then $q(is) \equiv 0$ for all $s \in [-i + \epsilon^{4/3}\rho_0, 0]$. Conversely, if $q(s_1) \neq 0$ for $s_1 \in [-1 + \epsilon^{4/3}\rho_0, -b)$, then F cannot satisfy symmetry condition: $\operatorname{Im} F = 0$ on $\{\operatorname{Re} \xi = 0\} \cap \mathcal{R}$.*

Proof. By equation (4.31)

$$q(s) = c_1 \exp\left\{\frac{1}{\epsilon} \int_{s_1}^s L(it) dt\right\} \{1 + o(1)\}$$

hence if $q(is_1) = 0$, then $c_1 = 0$. Conversely, if $q(is_1) \neq 0$, then $c_1 \neq 0$. Hence F cannot satisfy symmetry condition $\text{Im } F = 0$ on $\{\text{Re } \xi = 0\} \cap \mathcal{R}$. \square

Theorem 4.29. *Assume $S(a_n) = 0$, but $S'(a_n) \neq 0$, then for small enough ϵ and large enough ρ_0 , there is analytic function $\beta(\epsilon^{2/3})$ such that $\lim_{\epsilon \rightarrow 0} \beta(\epsilon^{2/3}) = a_n$, and if λ satisfies (1.28), then $\text{Im } F(\xi) = 0$ on $\{\xi = i\nu\} \cap \mathcal{R}$.*

Proof. For fixed large enough $\rho_0 > 0$, $S'(a_n) \neq 0$ implies

$$(4.32) \quad \frac{\partial \text{Im } \psi_0}{\partial a}(\rho_0, a_n) \neq 0,$$

Using Theorem 4.25, (4.32) and implicit function theorem, there exists analytic function $\beta(\epsilon^{2/3})$ so that $\text{Im } \psi(\rho_0, \epsilon, \beta(\epsilon^{2/3})) = 0$. This implies that $q(i\nu)$ is zero at some point in $[-i + i\rho_0^{4/3}, -bi)$. Then using Lemma 4.28, we complete the proof. \square

Proof of Theorem 1.12: If λ satisfies restriction (1.31), from Theorem 4.29 and Theorem 2.39, $F(\xi)$ is solution of Finger Problem.

5. CONCLUSION AND DISCUSSION

In this paper we are concerned about existence and selection of steadily translating symmetric finger solutions in a Hele-Shaw cell by small but non-zero kinetic undercooling ϵ^2 . We rigorously conclude that for relative finger width λ in the range $[\frac{1}{2}, \lambda_m]$, with $\lambda_m - \frac{1}{2}$ small, symmetric finger solutions exist in the asymptotic limit of undercooling $\epsilon^2 \rightarrow 0$ if the Stokes constant for a relatively simple nonlinear differential equation is zero. This Stokes constant S depends on the parameter $a \equiv \frac{2\lambda-1}{(1-\lambda)}\epsilon^{-\frac{4}{3}}$ and earlier calculations [3] have shown this to be zero for a discrete set of values of a . While this result is similar to that obtained in [22] previously for Saffman-Taylor fingers by surface tension, the analysis for the problem with kinetic undercooling exhibits a number of subtleties as pointed out by Chapman and King [3]. The main subtlety is the behavior of Stokes lines at the finger tip, which leads to existence of possible non-analytic fingers by kinetic undercooling, while previous results [20] show Saffman-Taylor fingers by surface tension must be analytic. Results in this paper are consistent with recent numerical corner-free finger solutions obtained in [8].

6. APPENDIX A: PROPERTIES OF THE FUNCTION $P(\xi)$

We look into some properties of $P(\xi)$ which is defined by (2.36) and (2.47).

Lemma 6.1. *Re $P(\xi)$ increases along negative Re ξ axis $(-\infty, 0)$ with Re $P(-\infty) = \text{constant} \neq 0$ and $C_1|t - 2i|^{-2} \leq |\frac{d}{dt} \text{Re } P(t)| \leq \frac{C_2}{\nu}|t - 2i|^{-2}$, for $t \in (-\infty, -\nu)$ where C_1 and C_2 are positive constants, independent of ϵ and ν .*

Proof. Using (2.36) and (2.47), we obtain

$$\text{Re } P(\xi) = -\gamma \int_{-\nu}^{\xi} \frac{(s^2 + \gamma^2)^{1/2}}{s(s^2 + 1)} ds + \text{constant}.$$

The lemma follows from the above equation. \square

Lemma 6.2. *Re $P(\xi)$ increases monotonically on imaginary ξ axis from $-ib$ to 0 where $0 < b < \min\{1, \gamma\}$.*

Proof. Using (2.36) and (2.47), we obtain for $\xi = i\eta$

$$\operatorname{Re} P(\xi) = - \int_{-b}^{\eta} \frac{(\gamma + t)(\gamma^2 - t^2)^{1/2}}{t(1 - t^2)} + \text{constant}.$$

The lemma follows from the above equation. \square

Lemma 6.3. *There exists a constant R independent of ϵ so that for $|\xi| \geq R$, $\operatorname{Re} P(t)$ increases with decreasing s along any ray $r = \{t : t = \xi - se^{i\varphi}, 0 < s < \infty, 0 \leq \varphi < \frac{\pi}{2}\}$ in \mathcal{R} from ξ to $\xi - \infty e^{i\varphi}$ and $C_1|t - 2i|^{-2} \leq \left|\frac{d}{ds} \operatorname{Re} P(t(s))\right| \leq C_2|t - 2i|^{-2}$, where C_1 and C_2 are constants, independent of ϵ , with $C_1 > 0$.*

Proof. Using (2.36) and (2.47), we obtain for $|\xi| \rightarrow \infty$, $P(\xi) \sim -i \log \xi$, hence the lemma follows. \square

Corollary 6.4. *On line segment r_{u_1} with decreasing s , $\operatorname{Re} P$ increases with $\frac{d}{ds} \operatorname{Re} P(t(s)) > \frac{C}{|t(s) - 2i|^2} > 0$ for constant C independent of ϵ and λ when the latter is restricted to a compact subset of $(0, 1)$.*

Proof. This follows very simply from the previous lemma. \square

Lemma 6.5. *There exists a constant R independent of ϵ so that for $|\xi| \geq R$, $\operatorname{Re} P(t)$ increases with increasing s along any arc $r = \{t : t = |\xi|e^{is}, \pi/2 < s < 3\pi/2\}$ in \mathcal{R} .*

Proof. Using (2.36) and (2.47), we obtain for $|\xi| \rightarrow \infty$, $P(\xi) \sim -i \log \xi$, hence the lemma follows. \square

Lemma 6.6. *For λ in a compact subset of $(0, 1)$ and for R independent of ϵ , consider the line segment*

$$t = i\nu_1 - \nu_1 - se^{i\phi}, \quad 0 \leq s \leq R$$

with $\nu_1 > 0$. Then, there exists real ν_1 , ϕ sufficiently small in absolute value and depending only on R so that on this line segment

$$-\frac{d}{ds} \operatorname{Re} P(t(s)) > C > 0$$

where C is independent of ϵ .

Proof. Note for $\nu_1 = 0$ and $\phi = 0$, result holds from Lemma 5.1, with $C = C_1$ only depending on the lower bound for ν . Since $\frac{d}{ds} \operatorname{Re} P(t(s))$ is clearly a continuous function of ϕ and ν_1 , and uniformly continuous for s restricted in a compact set, it follows that there exists ϕ and ν_1 small enough that

$$-\frac{d}{ds} \operatorname{Re} P(t(s)) > \frac{C_1}{2} = C > 0$$

where C is only dependent on ν_0 \square

Corollary 6.7. For small enough $\nu_1 > 0$, on line segment $r_{u,2}$, parametrized by arclength s increasing towards R ,

$$-\frac{d}{ds} \operatorname{Re} P(t(s)) > C > 0$$

where constant C is independent of ϵ and λ .

Proof. For $r_{u,2}$, we use previous Lemma 5.6 with $\phi = 0$ to obtain desired result. \square

Lemma 6.8. *There exists sufficiently small $\nu_1 > 0$ independent of ϵ so that $\frac{d}{ds} [\operatorname{Re} P(t(s))] \geq C > 0$ on the parametrized straight lines $\{t(s) = -\nu_1 + se^{-i\frac{\pi}{6}}, 0 \leq s \leq 2\sqrt{3}\nu_1/3\}$ and $\{t(s) = -\nu_1 + se^{i\frac{\pi}{6}}, 0 \leq s \leq 2\sqrt{3}\nu_1/3\}$, C is some constant independent of ϵ and ν_1 .*

Proof. Note that $\tilde{Q}(\xi) \sim -\frac{\gamma^2}{\xi}$ near $\xi = 0$, hence $P(\xi) \sim -\gamma^2 \log \xi$ which implies the lemma. \square

Lemma 6.9. *There exist $0 < b < 1$ and $0 < \alpha_0 < \pi/2$ so that $\operatorname{Re} P(t)$ is decreasing along ray $r_l = \{\xi = -bi + se^{i(\pi+\alpha_0)}, 0 \leq s < \infty\}$.*

Proof. We want to show that

$$(6.1) \quad \frac{d}{ds} \operatorname{Re} P(\xi(s)) = \operatorname{Re}\{P'(\xi)e^{i(\pi+\alpha_0)}\} < 0 \text{ on } r_l.$$

Note:

$$(6.2) \quad P'(\xi) = i \frac{(\xi + i\gamma)}{\xi} \frac{((-i\gamma + \xi)(\gamma + \xi))^{1/2}}{(\xi + i)(\xi - i)};$$

so

$$(6.3) \quad \arg(P'(\xi)e^{i(\pi+\alpha_0)}) = \frac{3\pi}{2} + \alpha_0 - (\arg \xi(s) - \arg(\xi + i\gamma)) \\ - [(\arg(\xi + i) + \arg(\xi - i)) - \frac{1}{2}(\arg(\xi + i) + \arg(\xi - i))] \in [\pi, \frac{3\pi}{2} - \frac{\alpha_0}{2}]$$

which leads to (6.1). \square

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